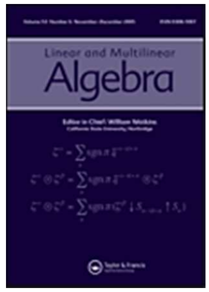

Linear maps preserving rank of tensor products of matrices

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LINEAR MAPS PRESERVING RANK OF TENSOR PRODUCTS OF MATRICES

BAODONG ZHENG, JINLI XU*, AND AJDA FOŠNER

Abstract. Let M_n be the algebra of all $n \times n$ complex matrices. We characterize linear maps $\phi : \mathcal{M}_{m_1 \cdots m_l} \to \mathcal{M}_{m_1 \cdots m_l}$ satisfying

$$\text{rank}\,(\phi\,(A_1 \otimes \cdots \otimes A_l)) = \text{rank}\,(A_1 \otimes \cdots \otimes A_l)$$

for all $A_k \in M_{m_k}$, $k = 1, \ldots, l$.

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Key words: Linear preserver, tensor product, rank of a matrix.

1. Introduction and the Main Theorem

Let M_n be the set of all $n \times n$ complex matrices. In quantum physics, quantum states of a system with n physical states are represented as density matrices, i.e., positive semi-definite matrices with trace one. Suppose $A \in M_m$ and $B \in M_n$ are the states of two quantum systems. Then their tensor (Kronecker) product $A \otimes B \in M_m \otimes M_n$ will be the state in the (bipartite) joint system. Density matrices in $M_m \otimes M_n \equiv M_{mn}$ that can be written as a convex combination of product states are separable states. It is easy to see that $S \in M_{mn}$ is separable if and only if it is a convex combination of $P_1 \otimes P_2$, where $P_1 \in M_m$ and $P_2 \in M_n$ are rank one orthogonal projections. In [6] it was shown that a linear map sending the set of separable states onto itself has a very nice structure. Namely, it has the form

$$A \otimes B \mapsto \varphi_1(A) \otimes \varphi_2(B)$$

or $m = n$ and

$$A \otimes B \mapsto \varphi_2(B) \otimes \varphi_1(A),$$

where φ_j, $j = 1, 2$, has the form $X \mapsto U_j X U_j^*$ or $X \mapsto U_j X^t U_j^*$ for some unitary $U_1 \in M_m$ and $U_2 \in M_n$. Here, Y^t denotes the transpose of a square matrix Y and Y^* the conjugate transpose of Y.

Motivated by the above observations, we characterize linear maps on M_{mn} which preserve rank of tensor products of matrices. In particular, we show that such a map has the form

$$A \otimes B \mapsto U(\tau_1(A) \otimes \tau_2(B))V$$

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for some invertible $U, V \in M_{mn}$, where τ_k, $k = 1, 2$, is either the identity map $X \mapsto X$ or the transposition map $X \mapsto X^t$. More generally, we consider linear maps preserving rank of tensor products of matrices on multipartite systems $M_{m_1} \otimes \cdots \otimes M_{m_l} = M_{m_1 \cdots m_l}$, $l \geq 2$.

The research on linear preserver problems has a long history. One may see [8] and its references for results on linear preserver problems. Furthermore, for the background on rank preserver problems (without the tensor structure) we refer the reader to [10, 12, 14]. It is well-known that a map $\phi : M_n \to M_n$ satisfies

$$\operatorname{rank}(\phi(A)) = \operatorname{rank}(A)$$

for all $A \in M_n$ if and only there exist invertible matrices $U, V \in M_n$ such that either

$$\phi(A) = UAV, \qquad A \in M_n,$$

or

$$\phi(A) = UA^tV, \qquad A \in M_n.$$

One can also see, for example, [1, 11] for results on rank preservers on tensor products of matrices. Moreover, for some recent research on linear preserver problems on tensor spaces arising in quantum information science we refer the reader to [2, 3, 4, 5, 6, 7, 9]. Continuing this line of investigations, we observe the structure of rank preservers on multipartite systems.

Before writing our main theorem, let us introduce some basic definitions and fix the notation. First of all, throughout the paper, l and $n, m_1, \ldots, m_l \geq 2$ are positive integers with $r = n - m_1 \cdots m_l \geq 0$. For an integer k, I_k denotes the $k \times k$ identity matrix, 0_k the $k \times k$ zero matrix, and $E_{ij}^{(k)}$, $1 \leq i, j \leq k$, the $k \times k$ matrix whose all entries are equal to zero except for the (i, j)-th entry which is equal to one. As usual, we use the notation $\operatorname{Diag}(a_1, \ldots, a_k)$ to denote the $k \times k$ diagonal matrix with diagonal entries $a_1, \ldots, a_k$.

We say that a linear map $\phi : M_{m_1 \cdots m_l} \to M_n$ preserves rank of tensor products of matrices if ϕ satisfies

$$\operatorname{rank}\left(\phi\left(A_1 \otimes \cdots \otimes A_l\right)\right) = \operatorname{rank}\left(A_1 \otimes \cdots \otimes A_l\right) \tag{1}$$

for all $A_k \in M_{m_k}$, $k = 1, \ldots, l$. We call a linear map π on $M_{m_1 \cdots m_l}$ canonical, if

$$\pi\left(A_1 \otimes \cdots \otimes A_l\right) = \tau_1\left(A_1\right) \otimes \cdots \otimes \tau_l\left(A_l\right)$$

for all $A_k \in M_{m_k}$, $k = 1, \ldots, l$, where $\tau_k : M_{m_k} \to M_{m_k}$, $k = 1, \ldots, l$, is either the identity map $X \mapsto X$ or the transposition map $X \mapsto X^t$. In this case we write $\pi = \tau_1 \otimes \cdots \otimes \tau_l$.

Our main result reads as follows.

Main Theorem. *A linear map $\phi : M_{m_1\cdots m_l} \to M_n$ preserves rank of tensor products of matrices if and only if there exist invertible matrices $U, V \in M_n$ and a canonical map π on $M_{m_1\cdots m_l}$ such that*

$$\phi(A_1 \otimes \cdots \otimes A_l) = U\left(\pi(A_1 \otimes \cdots \otimes A_l) \oplus 0_{n-m_1\cdots m_l}\right)V. \tag{2}$$

for all $A_k \in M_{m_k}$, $k = 1, \ldots, l$. In particular, if $n = m_1 \cdots m_l$, then

$$\phi(A_1 \otimes \cdots \otimes A_l) = U\pi(A_1 \otimes \cdots \otimes A_l)V$$

for all $A_1 \otimes \cdots \otimes A_l \in M_{m_1\cdots m_l}$.

Remark 1.1. If $n < m_1 \cdots m_l$ and $\phi : M_{m_1\cdots m_l} \to M_n$ is a linear map, then

$$\text{rank}\,(I_{m_1\cdots m_l}) = m_1 \cdots m_l$$

and

$$\text{rank}\,(\phi(I_{m_1\cdots m_l})) \le n < m_1 \cdots m_l.$$

So, ϕ does not satisfy (1) for all $A_1 \otimes \cdots \otimes A_l \in M_{m_1\cdots m_l}$.

Remark 1.2. Let us point out that in our Main Theorem we characterize maps which preserve rank of tensor products of matrices and that the resulting map may not preserve rank of all matrices in $M_{m_1\cdots m_l}$. More precisely, if $l > 1$, then the map ϕ of the form (2) may not satisfy $\text{rank}\,(X) = \text{rank}\,(\phi(X))$ for all $X \in M_{m_1\cdots m_l}$. For example, let $n = m_1 \cdots m_l$ and

$$\phi : A_1 \otimes A_2 \cdots \otimes A_l \mapsto A_1^t \otimes A_2 \cdots \otimes A_l$$

Denote

$$X = E_{11}^{(m_1)} \otimes E_{11}^{(m_2)} + E_{12}^{(m_1)} \otimes E_{12}^{(m_2)} + E_{21}^{(m_1)} \otimes E_{21}^{(m_2)} + E_{22}^{(m_1)} \otimes E_{22}^{(m_2)}$$

and

$$Y = E_{11}^{(m_1)} \otimes E_{11}^{(m_2)} + E_{21}^{(m_1)} \otimes E_{12}^{(m_2)} + E_{12}^{(m_1)} \otimes E_{21}^{(m_2)} + E_{22}^{(m_1)} \otimes E_{22}^{(m_2)}.$$

Of course, $X, Y \in \mathcal{M}_{m_1 m_2}$ and

$$\phi(X \otimes I_{m_3\cdots m_l}) = Y \otimes I_{m_3\cdots m_l}.$$

It is also easy to see that

$$\text{rank}\,(X \otimes I_{m_3\cdots m_l}) \ne \text{rank}\,(\phi(X \otimes I_{m_3\cdots m_l})) = \text{rank}\,(Y \otimes I_{m_3\cdots m_l}).$$

In a special case, when $l = 2$ and $m_1 = m_2 = 2$, we have

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Y = \phi(X) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. Lemmas

Before proving our main result, let us write several lemmas which we will need in the sequel. The first one is a well-known theorem from the paper [13].

Lemma 2.1. [13, Theorem 12] *Let m, n, s be positive integers with $ms \leq n$. Suppose that $X_k \in M_n$, $k = 1, \ldots, m$, are complex matrices satisfying*

$$\operatorname{rank}(X_k) = s \quad \textit{and} \quad \operatorname{rank}\left(\sum_{k=1}^{m} X_k\right) = ms.$$

Then there exist invertible matrices $U, V \in M_n$ such that

$$X_k = U\left(\left(E_{kk}^{(m)} \otimes I_s\right) \oplus 0_{n-ms}\right) V, \qquad k = 1, \ldots, m.$$

Lemma 2.2. *Let m_1, m_2, n be positive integers with $k \leq m_1$ and $m_1 m_2 \leq n$. Suppose that $A \in M_{m_2}$ is an invertible matrix and $X \in M_n$ such that*

$$\operatorname{rank}\left(\left(\lambda E_{kk}^{(m_1)} \otimes A\right) \oplus 0_{n-m_1 m_2} + X\right) = m_2$$

for all $\lambda \in \mathbb{C}$. Then X is of the form

$$X = \begin{bmatrix} 0_{(k-1)m_2} & X_{12} & 0 \\ X_{21} & X_{22} & X_{23} \\ 0 & X_{32} & 0_{n-km_2} \end{bmatrix},$$

where $X_{22} \in M_{m_2}$.

Proof. Let

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix},$$

with $X_{11} \in M_{(k-1)m_2}$ and $X_{22} \in M_{m_2}$. Then

$$\operatorname{rank}\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & \lambda A + X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} = m_1$$

for all $\lambda \in \mathbb{C}$. Since A is invertible there exist infinitely many complex numbers λ such that $\det(\lambda A + X_{22}) \neq 0$ (here, $\det(\lambda A + X_{22})$ denotes the determinant of a matrix $\lambda A + X_{22}$). For those scalars λ, rank of the matrix

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & \lambda A + X_{22} & X_{23} \\ X_{31} - X_{32}(\lambda A + X_{22})^{-1} X_{21} & 0 & X_{33} - X_{32}(\lambda A + X_{22})^{-1} X_{21} \end{bmatrix}$$

is equal to m_2 and, hence,

$$X_{33} = X_{32}(\lambda A + X_{22})^{-1} X_{21}.$$

Let $\operatorname{adj}(\lambda A + X_{22})$ be the adjoint matrix of $\lambda A + X_{22}$. Then

$$(\det(\lambda A + X_{22})) X_{33} = X_{32}(\operatorname{adj}(\lambda A + X_{22})) X_{21}.$$

Recall that $\det(\lambda A + X_{22})$ is a polynomial of λ with degree m_2, and, thus, each entry of the left matrix above is either a polynomial of λ with degree m_2 or zero. On the other hand, $\operatorname{adj}(\lambda A + X_{22})$ is a $m_2 \times m_2$ matrix formed by taking the transpose of the cofactor matrix of a given matrix $\lambda A + X_{22}$. In particular, its entries are polynomials of λ with degree at most $m_2 - 1$. So, each position of the right matrix above is a polynomial of λ with degree at most $m_2 - 1$. This yields that $X_{33} = 0$. Similarly, $X_{11} = 0, X_{13} = 0, X_{31} = 0$. □

Lemma 2.3. *Let m_1, m_2, n, i, j be positive integers with $i \neq j \leq m_1$ and $m_1 m_2 \leq n$. Suppose that $A_1, A_2 \in M_{m_2}$ are invertible matrices and $X \in M_n$ satisfying*

$$\operatorname{rank}\left(\left(\lambda E_{ii}^{(m_1)} \otimes A_1\right) \oplus 0_{n-m_1m_2} + X\right) = m_2,$$
$$\operatorname{rank}\left(\left(\lambda E_{jj}^{(m_1)} \otimes A_2\right) \oplus 0_{n-m_1m_2} + X\right) = m_2$$

for all $\lambda \in \mathbb{C}$. Then there exist $Y, Z \in M_{m_2}$ such that

$$X = \left(E_{ij}^{(m_1)} \otimes Y + E_{ji}^{(m_1)} \otimes Z\right) \oplus 0_{n-m_1m_2}.$$

Proof. Applying lemma 2.2, the result follows. □

Lemma 2.4. *Let π be a canonical map on $M_{m_1\cdots m_l}$ and $U, V \in M_{m_1\cdots m_l}$. If*

$$U\pi(A_1 \otimes \cdots \otimes A_l)V = 0$$

for all $A_k \in M_{m_k}$, $k = 1, \ldots, l$, then $U = 0$ or $V = 0$.

Proof. By the assumption, it is easy to see that $UXV = 0$ for all $X \in M_{m_1\cdots m_l}$ and, thus, either $U = 0$ or $V = 0$, as desired. □

Recall that if a matrix $U \in M_{kh}$ commutes with $I_k \otimes S$ for all real symmetric $S \in M_h$, then U has the form $W \otimes I_h$ with $W \in M_k$. This simple observation will be used in the proof of our last lemma.

Lemma 2.5. *Let π_1, π_2 be canonical maps on $M_{m_1\cdots m_k}$ and $U, V \in M_{m_1\cdots m_l}$ invertible matrices. If*

$$\pi_1(A_1 \otimes \cdots \otimes A_l)U = V\pi_2(A_1 \otimes \cdots \otimes A_l)$$

for all $A_k \in M_{m_k}$, $k = 1, \ldots, l$, then $\pi_1 = \pi_2$ and $U = V = \lambda I_{m_1\cdots m_l}$ for some nonzero scalar $\lambda \in \mathbb{C}$.

Proof. Since π_1, π_2 are canonical maps on $M_{m_1\cdots m_l}$, we can write

$$\pi_1 = \tau_1 \otimes \cdots \otimes \tau_l \quad \text{and} \quad \pi_2 = \eta_1 \otimes \cdots \otimes \eta_l,$$

where each of the maps τ_k and η_k, $k = 1, \ldots, l$, is either the identity map $X \mapsto X$ or the transposition map $X \mapsto X^t$.

Clearly, $\pi_i(I_{m_1\cdots m_l}) = I_{m_1\cdots m_l}$, $i = 1, 2$. Hence, $U = V$. To prove that U is a scalar multiple of the identity map we use the induction on l. The case $l = 1$ is obvious. So, let $l \geq 2$ and assume that the statement holds true for $l-1$. Note that for any real symmetric $S \in M_{m_l}$, we have $\pi_i(I_{m_1\cdots m_{l-1}} \otimes S) =$

$I_{m_1\cdots m_{l-1}} \otimes S$, $i = 1, 2$. This implies that U commutes with $I_{m_1\cdots m_{l-1}} \otimes S$ for all real symmetric $S \in M_{m_l}$, and, thus, $U = W \otimes I_{m_l}$ for some matrix $W \in M_{m_1\cdots m_{l-1}}$. Since U is invertible, W has to be invertible as well.

Let us define linear maps

$$\widetilde{\pi}_1 = \tau_1 \otimes \cdots \otimes \tau_{l-1} \quad \text{and} \quad \widetilde{\pi}_2 = \eta_1 \otimes \cdots \otimes \eta_{l-1}.$$

It is clear that $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ are canonical maps on $M_{m_1\cdots m_{l-1}}$. Let $Y = A_1 \otimes \cdots \otimes A_{l-1} \in M_{m_1\cdots m_{l-1}}$ be an arbitrary matrix. Then

$$(\widetilde{\pi}_1(Y)W) \otimes I_{m_l} = \pi_1(Y \otimes I_{m_l})U = U\pi_2(Y \otimes I_{m_l}) = (W\widetilde{\pi}_2(Y)) \otimes I_{m_l}$$

and, thus, $\widetilde{\pi}_1(Y)W = W\widetilde{\pi}_2(Y)$. Consequently, by the induction hypothesis, $W = \lambda I_{m_1\cdots m_{l-1}}$ for some nonzero scalar λ. Therefore, $U = \lambda I_{m_1\cdots m_l}$ and

$$\pi_1(X) = \lambda^{-1}\pi_1(X)U = \lambda^{-1}U\pi_2(X) = \pi_2(X)$$

for all $X = A_1 \otimes \cdots \otimes A_l \in M_{m_1\cdots m_l}$. □

3. Proof of the Main Theorem

Since the sufficiency part of the Main Theorem is clear, we consider only the necessity part. So, throughout this section, we assume that $\phi : \mathcal{M}_{m_1\cdots m_l} \to \mathcal{M}_n$ is a linear map which satisfies (1). Using the induction on l, we prove that ϕ is of the form (2).

The case $l = 1$ is just the corollary of Theorem 2.1 in [10]. So, assume that $l > 1$ and that the expected result holds true for $l - 1$.

Claim 1. If $F_1, \ldots, F_{m_1} \in M_{m_1}$ are rank one matrices with

$$\operatorname{rank}\left(\sum_{k=1}^{m_1} F_k\right) = m_1,$$

then there exist invertible matrices $U, V \in M_n$ such that for all $k = 1, \ldots, m_1$,

$$\phi(F_k \otimes X) = U\left(\left(E_{kk}^{(m_1)} \otimes \pi_k(X)\right) \oplus 0_r\right)V$$

for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2\cdots m_l}$, where π_k are canonical maps on $M_{m_2\cdots m_l}$.

Proof. Since

$$\operatorname{rank}(F_k \otimes I_{m_2\cdots m_l}) = m_2 \cdots m_l$$

and

$$\operatorname{rank}\left(\sum_{k=1}^{m_1} (F_k \otimes I_{m_2\cdots m_l})\right) = m_1 m_2 \cdots m_l$$

it follows that

$$\operatorname{rank}(\phi(F_k \otimes I_{m_2\cdots m_l})) = m_2 \cdots m_l$$

and
$$\operatorname{rank}\left(\sum_{k=1}^{m_1} \phi\left(F_k \otimes I_{m_2\cdots m_l}\right)\right) = m_1 m_2 \cdots m_l.$$
By Lemma 2.1, there exist invertible matrices $U, V \in M_n$ such that
$$\phi\left(F_k \otimes I_{m_2\cdots m_l}\right) = U\left(\left(E_{kk}^{(m_1)} \otimes I_{m_2\cdots m_l}\right) \oplus 0_r\right) V, \qquad k = 1, \ldots, m_1.$$
After composing ϕ by the linear transformation $X \mapsto U^{-1}XV^{-1}$, we may assume that
$$\phi\left(F_k \otimes I_{m_2\cdots m_l}\right) = \left(E_{kk}^{(m_1)} \otimes I_{m_2\cdots m_l}\right) \oplus 0_r, \qquad k = 1, \ldots, m_1. \tag{3}$$
For $k = 1, \ldots, m_1$, let us define maps $L_k : M_{m_2\cdots m_l} \to M_n$ by
$$L_k(X) = \phi\left(F_k \otimes X\right).$$
It follows from the property of ϕ that L_k are linear maps which preserve rank of tensor products of matrices. Thus, applying the induction hypothesis on L_k, we conclude that there exist invertible matrices $R_k, S_k \in M_n$ such that
$$\phi\left(F_k \otimes X\right) = R_k\left(E_{kk}^{(m_1)} \otimes \pi_k\left(X\right) \oplus 0_r\right) S_k \tag{4}$$
for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2\cdots m_l}$, where π_k is a canonical map on $M_{m_2\cdots m_l}$. It follows form (3) and (4) that
$$\left(E_{kk}^{(m_1)} \otimes I_{m_2\cdots m_l}\right) \oplus 0_r = R_k\left(\left(E_{kk}^{(m_1)} \otimes I_{m_2\cdots m_l}\right) \oplus 0_r\right) S_k. \tag{5}$$
Set
$$R_k = \begin{bmatrix} R_{11}^k & R_{12}^k & R_{13}^k \\ R_{21}^k & R_{22}^k & R_{23}^k \\ R_{31}^k & R_{32}^k & R_{33}^k \end{bmatrix}, \quad S_k = \begin{bmatrix} S_{11}^k & S_{12}^k & S_{13}^k \\ S_{21}^k & S_{22}^k & S_{23}^k \\ S_{31}^k & S_{32}^k & S_{33}^k \end{bmatrix},$$
where $R_{11}^k, S_{11}^k \in M_{(k-1)m_2\cdots m_l}$ and $R_{22}^k, S_{22}^k \in M_{m_2\cdots m_l}$. Using (5), we obtain
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{m_2\cdots m_l} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{11}^k & R_{12}^k & R_{13}^k \\ R_{21}^k & R_{22}^k & R_{23}^k \\ R_{31}^k & R_{32}^k & R_{33}^k \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{m_2\cdots m_l} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_{11}^k & S_{12}^k & S_{13}^k \\ S_{21}^k & S_{22}^k & S_{23}^k \\ S_{31}^k & S_{32}^k & S_{33}^k \end{bmatrix}.$$
More precisely,
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{m_2\cdots m_l} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{12}^k S_{21}^k & R_{12}^k S_{22}^k & R_{12}^k S_{23}^k \\ R_{22}^k S_{21}^k & R_{22}^k S_{22}^k & R_{22}^k S_{23}^k \\ R_{32}^k S_{21}^k & R_{32}^k S_{22}^k & R_{32}^k S_{23}^k \end{bmatrix}.$$
Since $R_{22}^k S_{22}^k = I_{m_2\cdots m_l}$ it follows that $R_{12}^k = R_{32}^k = S_{21}^k = S_{23}^k = 0$. Therefore,
$$R_k = \begin{bmatrix} R_{11}^k & 0 & R_{13}^k \\ R_{21}^k & R_{22}^k & R_{23}^k \\ R_{31}^k & 0 & R_{33}^k \end{bmatrix}, \quad S_k = \begin{bmatrix} S_{11}^k & S_{12}^k & S_{13}^k \\ 0 & S_{22}^k & 0 \\ S_{31}^k & S_{32}^k & S_{33}^k \end{bmatrix}.$$

This, together with (4), yields

$$\phi\left(F_k \otimes X\right) = \left(E_{kk}^{(m_1)} \otimes R_{22}^k \pi_k\left(X\right) S_{22}^k\right) \oplus 0_r$$

for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2 \cdots m_l}$. Set $T = \mathrm{Diag}\left(R_{22}^1, \ldots, R_{22}^{m_1}, I_r\right)$. Then for all $k = 1, \ldots, m_1$ and $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2 \cdots m_l}$,

$$\phi\left(F_k \otimes X\right) = T\left(\left(E_{kk}^{(m_1)} \otimes \pi_k\left(X\right)\right) \oplus 0_r\right) T^{-1}.$$

□

According to Claim 1, we may assume that for all $k = 1, \ldots, m_1$ and $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2 \cdots m_l}$,

$$\phi\left(E_{kk}^{(m_1)} \otimes X\right) = \left(E_{kk}^{(m_1)} \otimes \pi_k\left(X\right)\right) \oplus 0_r, \tag{6}$$

where π_k are canonical maps on $M_{m_2 \cdots m_l}$.

Claim 2. We prove that $\pi_i = \pi_j = \pi$ for all $1 \leq i < j \leq m_1$. Moreover, there exist scalars $\lambda_{ij} \neq 0$ such that

$$\phi\left(E_{ij}^{(m_1)} \otimes X\right) = \left(\lambda_{ij}\tau_{ij}\left(E_{ij}^{(m_1)}\right) \otimes \pi\left(X\right)\right) \oplus 0_r, \tag{7}$$

for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2 \cdots m_l}$, where τ_{ij} is either the identity map $Y \mapsto Y$ or the transposition map $Y \mapsto Y^t$.

Proof. For the sake of the simplicity, let us assume that $i = 1$ and $j = 2$. Denote $F_1 = E_{11}^{(m_1)} + E_{12}^{(m_1)}$, $F_2 = E_{22}^{(m_1)} + E_{12}^{(m_1)}$, and $F_k = E_{kk}^{(m_1)}$ for $k = 3, \ldots, m_1$. Then, clearly, matrices F_k, $k = 1, \ldots, m_1$, satisfy the assumptions in Claim 1 and, thus, there exist invertible matrices $U, V \in M_n$ such that for all $k = 1, \ldots, m_1$ and $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2 \cdots m_l}$,

$$\phi\left(F_k \otimes X\right) = U\left(\left(E_{kk}^{(m_1)} \otimes \widetilde{\pi}_k\left(X\right)\right) \oplus 0_r\right) V, \tag{8}$$

where $\widetilde{\pi}_k$ are canonical maps on $M_{m_2 \cdots m_l}$.

Set

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} \text{ and } V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix},$$

where $U_{11}, U_{22}, V_{11}, V_{22} \in M_{m_2 \cdots m_l}$ and $U_{33}, V_{33} \in M_{n-2m_2 \cdots m_l}$. Then, using (6) and (8), we obtain

$$\phi\left(E_{12}^{(m_1)} \otimes X\right) = \phi\left(F_1 \otimes X\right) - \phi\left(E_{11} \otimes X\right) = \tag{9}$$

$$= \begin{bmatrix} U_{11}\widetilde{\pi}_1\left(X\right)V_{11} - \pi_1\left(X\right) & U_{11}\widetilde{\pi}_1\left(X\right)V_{12} & U_{11}\widetilde{\pi}_1\left(X\right)V_{13} \\ U_{21}\widetilde{\pi}_1\left(X\right)V_{11} & U_{21}\widetilde{\pi}_1\left(X\right)V_{12} & U_{21}\widetilde{\pi}_1\left(X\right)V_{13} \\ U_{31}\widetilde{\pi}_1\left(X\right)V_{11} & U_{31}\widetilde{\pi}_1\left(X\right)V_{12} & U_{31}\widetilde{\pi}_1\left(X\right)V_{13} \end{bmatrix}$$

and

$$\phi\left(E_{12}^{(m_1)} \otimes X\right) = \phi\left(F_2 \otimes X\right) - \phi\left(E_{22} \otimes X\right) = \tag{10}$$

$$= \begin{bmatrix} U_{12}\widetilde{\pi}_2(X)V_{21} & U_{12}\widetilde{\pi}_2(X)V_{22} & U_{12}\widetilde{\pi}_2(X)V_{23} \\ U_{22}\widetilde{\pi}_2(X)V_{21} & U_{22}\widetilde{\pi}_2(X)V_{22} - \pi_2(X) & U_{22}\widetilde{\pi}_2(X)V_{23} \\ U_{32}\widetilde{\pi}_2(X)V_{21} & U_{32}\widetilde{\pi}_2(X)V_{22} & U_{32}\widetilde{\pi}_2(X)V_{23} \end{bmatrix}$$

for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2\cdots m_l}$.

Note that for any invertible matrix $G_k \in M_{m_k}$, $k = 2, \ldots, l$, and any scalar $\varepsilon \in \mathbb{C}$, matrices $(E_{11}^{(m_1)} + \varepsilon E_{12}^{(m_1)}) \otimes G_2 \otimes \cdots \otimes G_l$ and $(E_{22}^{(m_1)} + \varepsilon E_{12}^{(m_1)}) \otimes G_2 \otimes \cdots \otimes G_l$ are of rank $m_2 \cdots m_l$. Therefore, using (6), we conclude that matrices

$$\left(E_{11}^{(m_1)} \otimes \pi_1(G_2 \otimes \cdots \otimes G_l)\right) \oplus 0_r + \varepsilon\phi\left(E_{12}^{(m_1)} \otimes G_2 \otimes \cdots \otimes G_l\right),$$

$$\left(E_{22}^{(m_1)} \otimes \pi_2(G_2 \otimes \cdots \otimes G_l)\right) \oplus 0_r + \varepsilon\phi\left(E_{12}^{(m_1)} \otimes G_2 \otimes \cdots \otimes G_l\right)$$

have rank $m_2 \cdots m_l$. This yields, by Lemma 2.3, that

$$\phi\left(E_{12}^{(m_1)} \otimes X\right) = \begin{bmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad X \in M_{m_2\cdots m_l}, \tag{11}$$

where $*$ stands for some $(m_2 \cdots m_l) \times (m_2 \cdots m_l)$ complex matrix. It follows from (9) and (11) that $U_{21}\widetilde{\pi}_1(X)V_{12} = 0$ for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2\cdots m_l}$ and, by Lemma 2.4, either $U_{21} = 0$ or $V_{12} = 0$.

Case I. Assume first that $U_{21} = 0$. Using (9), (10), and (11), we obtain

$$U_{11}\widetilde{\pi}_1(X)V_{11} = \pi_1(X) \quad \text{and} \quad U_{22}\widetilde{\pi}_2(X)V_{22} = \pi_2(X)$$

for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2\cdots m_l}$. Writing $I_{m_2\cdots m_l}$ instead of X in the above equalities, we get $U_{11}V_{11} = I_{m_2\cdots m_l}$ and $U_{22}V_{22} = I_{m_2\cdots m_l}$. Hence,

$$U_{11}\widetilde{\pi}_1(X) = \pi_1(X)V_{11}^{-1} \quad \text{and} \quad \widetilde{\pi}_2(X)V_{22} = U_{22}^{-1}\pi_2(X).$$

Using (9) and (10), we conclude that

$$\phi(E_{12}^{(m_1)} \otimes X) = \left(E_{12}^{(m_1)} \otimes \left(\pi_1(X)V_{11}^{-1}V_{12}\right)\right) \oplus 0_r$$

and

$$\phi(E_{12}^{(m_1)} \otimes X) = \left(E_{12}^{(m_1)} \otimes \left(U_{12}U_{22}^{-1}\pi_2(X)\right)\right) \oplus 0_r$$

and, hence,

$$\pi_1(X)V_{11}^{-1}V_{12} = U_{12}U_{22}^{-1}\pi_2(X)$$

for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2\cdots m_l}$. Since

$$\operatorname{rank}\left(\phi(E_{12}^{(m_1)} \otimes I_{m_2\cdots m_l})\right) = m_2 \cdots m_l$$

it follows that $V_{11}^{-1}V_{12}$ and $U_{12}U_{22}^{-1}$ are invertible matrices and, by Lemma 2.5, we have $\pi_1 = \pi_2$ and $V_{11}^{-1}V_{12} = \lambda_{12}I_{m_2\cdots m_l}$ for some $\lambda_{12} \neq 0$. Thus,

$$\phi(E_{12}^{(m_1)} \otimes X) = \left(\lambda_{12}E_{12}^{(m_1)} \otimes \pi_1(X)\right) \oplus 0_r$$

for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2\cdots m_l}$.

Case II. If $V_{12} = 0$, then, using a similar approach as in Case I, we show that there exists a nonzero scalar $\lambda_{12} \in \mathbb{C}$ such that

$$\phi(E_{12}^{(m_1)} \otimes X) = \left(\lambda_{12} E_{21}^{(m_1)} \otimes \pi_1(X)\right) \oplus 0_r$$

for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2 \cdots m_l}$. $\square$

Using the above observations, we complete the proof of our Main Theorem. First, by (6) and (7), we may assume that for all $1 \le i, j \le m_1$, there exist scalars $\lambda_{ij} \neq 0$ and a canonical map π on $M_{m_2 \cdots m_l}$ such that

$$\phi\left(E_{ij}^{(m_1)} \otimes X\right) = \left(\lambda_{ij}\tau_{ij}\left(E_{ij}^{(m_1)}\right) \otimes \pi(X)\right) \oplus 0_r \tag{12}$$

for all $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2 \cdots m_l}$, where τ_{ij} is either the identity map $Y \mapsto Y$ or the transposition map $Y \mapsto Y^t$.

According to (6), (12), and

$$\operatorname{rank}\left(E_{ii}^{(m_1)} + E_{ij}^{(m_1)} + E_{ji}^{(m_1)} + E_{jj}^{(m_1)}\right) \otimes I_{m_2 \cdots m_l} = m_2 \cdots m_l,$$

we conclude that

$$\operatorname{rank}\left(E_{ii}^{(m_1)} + \lambda_{ij}\tau_{ij}(E_{ij}^{(m_1)}) + \lambda_{ji}\tau_{ji}\left(E_{ji}^{(m_1)}\right) + E_{jj}^{(m_1)}\right) \otimes I_{m_2 \cdots m_l} = m_2 \cdots m_l.$$

This yields that $\lambda_{ij}\lambda_{ji} = 1$ and $\tau_{ij} = \tau_{ji}$ for all $1 \le i < j \le m_1$.

Assume for a moment that $m_1 \ge 3$ and that $1 \le i, j, k \le m_1$ are distinct positive integers. Then, according to (6), (12), and

$$\operatorname{rank}\left(E_{ii}^{(m_1)} + E_{ij}^{(m_1)} + E_{ik}^{(m_1)}\right) \otimes I_{m_2 \cdots m_l} = m_2 \cdots m_l,$$

$$\operatorname{rank}\left(E_{ij}^{(m_1)} + E_{ik}^{(m_1)} + E_{jj}^{(m_1)} + E_{jk}^{(m_1)}\right) \otimes I_{m_2 \cdots m_l} = m_2 \cdots m_l,$$

we obtain

$$\operatorname{rank}\left(E_{ii}^{(m_1)} + \lambda_{ij}\tau_{ij}\left(E_{ij}^{(m_1)}\right) + \lambda_{ik}\tau_{ik}\left(E_{ik}^{(m_1)}\right)\right) \otimes I_{m_2 \cdots m_l} = m_2 \cdots m_l$$

and

$$\operatorname{rank}\left(\lambda_{ij}\tau_{ij}(E_{ij}^{(m_1)}) + \lambda_{ik}\tau_{ik}\left(E_{ik}^{(m_1)}\right) + E_{jj}^{(m_1)} + \lambda_{jk}\tau_{jk}\left(E_{jk}^{(m_1)}\right)\right) \otimes I_{m_2 \cdots m_l} =$$
$$= m_2 \cdots m_l.$$

Therefore, $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$, $\tau_{ik} = \tau_{ij}$, and $\tau_{ij} = \tau_{jk}$. This yields that $\tau_{ij} = \tau_{12} = \tau$ for all $1 \le i < j \le m_1$. Now, writing

$$T = \left(\operatorname{Diag}\left(1, \lambda_{12}^{-1}, \ldots, \lambda_{1m_1}^{-1}\right) \otimes I_{m_2 \cdots m_l}\right) \oplus I_r,$$

we have

$$\phi(X_1 \otimes X) = T\left(\left(\tau(X_1) \otimes \pi(X)\right) \oplus 0_r\right) T^{-1}$$

for all $X_1 \in M_{m_1}$ and $X = X_2 \otimes \cdots \otimes X_l \in M_{m_2 \cdots m_l}$. The proof of the Main Theorem is completed.

Remark 3.1. Let $k = 1, \ldots, m_1 \cdots m_l$ and let us denote

$$R_k := \{A_1 \otimes \cdots \otimes A_l \in \mathcal{M}_{m_1 \cdots m_l} : \operatorname{rank}(A_1 \otimes \cdots \otimes A_l) = k\}.$$

In our Main Theorem we characterize maps ϕ on $M_{m_1 \cdots m_l}$ which leave all the sets R_k invariant, i.e.,

$$\phi(R_k) \subseteq R_k, \qquad k = 1, \ldots, m_1 \cdots m_l. \tag{13}$$

Here, it is also interesting to study linear maps ϕ such that (13) holds true just for some fixed k. For example, we would like to determine the structure of linear maps $\phi : M_{m_1 \cdots m_l} \to M_{m_1 \cdots m_l}$ such that $\phi(A_1 \otimes \cdots \otimes A_l)$ is a rank one matrix whenever matrices $A_i \in M_{m_k}$, $i = 1, \ldots, l$, have rank one, i.e., $\phi(R_1) \subseteq R_1$. Even for $l = 2$ this is still an open problem.

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