Amidakuji (ghost leg): From a simple game to research in different areas

Chi-Kwong Li
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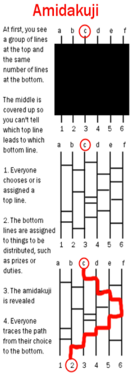
Amidakuji/Ghost Leg Drawing



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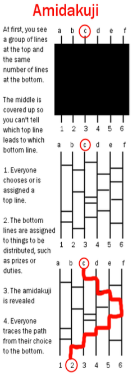

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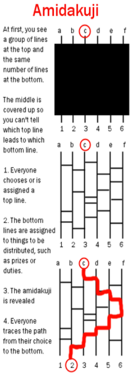

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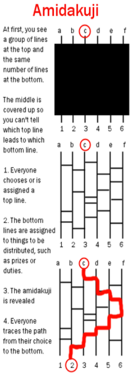

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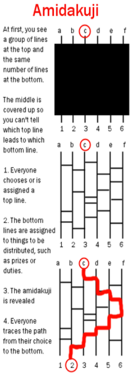

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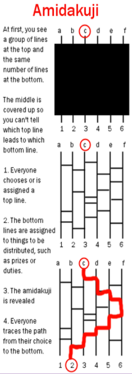

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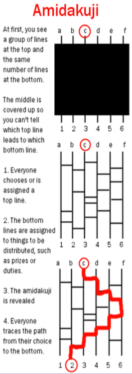

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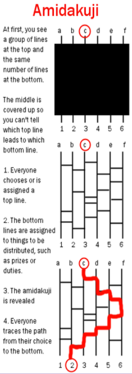

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- Can we get all possible job assignments?
- What is the minimum number of horizontal segments needed for a given job assignment?

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- An easy induction argument!

Bubble sort

- Regard the job assignment as a permutation (a seat reassignment)

$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

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Example For $\sigma = [5, 3, 1, 2, 4]$, total number of inversions is: $4 + 0 + 2 = 6$, and

$$\sigma \to [3, 5, 1, 2, 4] \to [3, 1, 5, 2, 4] \to [3, 1, 2, 5, 4]$$

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$$(n-1) + \cdots + 1 = n(n-1)/2 \text{ steps.}$$

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Example. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 9 & 6 & 7 & 1 & 8 & 2 \end{pmatrix} = (1,3,5,6,7)(2,4,9)(8).$

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Open problems

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- For a given $\sigma \in S_n$, find the smallest r such that σ is the product of r transpositions in G_m.
- Determine the optimal (smallest) $r^* = r^*(n, m)$ so that every $\sigma \in S_n$ is a product at most r^* transpositions in G_m.

We have the following list for $r^*(n,m)$ for S_n and $(i, i+\ell)$ with $\ell \le m$,

$n\backslash m$	1	2	3	4	5	6	7	8	9	10
2	1									
3	3	2								
4	6	4	3							
5	10	5	5	4						
6	15	[7]	6	6	5					
7	21	[10]	8	7	7	6				
8	28	[14]	[10]	9	8	8	7			
9	36	[16]	[11]	10	10	9	9	8		
10	45	[19]	[14]	[12]	11	11	10	10	9	
11	55	24?	18?	15?	13	12	12	11	11	10

where the entries marked by brackets are obtained by computer programming.

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- Diameter of the Cayley graph of S_n using L and S:

S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
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- Diameter of the Cayley graph of S_n using L, S, and $R = L^{-1} = (n, n-1, \ldots, 1)$:

S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
1	2	6	10	15	21	28	36	45	???

Computing

- Every orthogonal (unitary) matrix P can be written as the product of orthogonal matrices of the form

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For $n = 4$, we need

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

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- More on Quantum computing.
  It is of interest to decompose certain quantum gates into simpler quantum gates (CNOT gates).

Can you think of other connections!

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Thank you for your attention!