Quantum Mahjong

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- So, there are 2 possible winning states:

$$\rho_1 = E_{11} \otimes E_{11} \otimes E_{11} \otimes E_{22} \otimes E_{22} \text{ and } \rho_2 = E_{11} \otimes E_{11} \otimes E_{22} \otimes E_{22} \otimes E_{22}.$$

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- The states ρ_1, ρ_2 may appear with probabilities p and $1 - p$, say, controlled by the casino.
- We have to set up a positive operator valued measurement $\{M_1, M_2\}$ such that M_1, M_2 are positive semidefinite such that $M_1 + M_2 = I_N$ and maximize the expected winning probability:

$$q = \mathrm{tr}\,(pM_1\rho_1 + (1 - p)M_2\rho_2).$$

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- Because ρ_1 and ρ_2 are orthogonal, we can set $M_1 = \rho_1$ and $M_2 = I_N - \rho_1$ to get $q = 1$.

- However, the casino owner may use a different basis to represent ρ_2, i.e., change ρ_2 to $\hat{\rho}_2 = U^* \rho_2 U$ for some unitary matrix $U \in M_N$.

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- The winning hand will be $X_1X_2X_3X_jX_j$ for $j \in \{1,2,3\}$.
- So, there are three possible winning states $\rho_1, \rho_2, \rho_3 \in M_N$ with

$$\rho_j = E_{11} \otimes E_{22} \otimes E_{33} \otimes E_{jj} \otimes E_{jj},$$

corresponding to these patterns, say, with probability p_1, p_2, p_3, controlled by the casino.

- We have to set up a positive operator valued measurement $\{M_1, M_2, M_3\}$ such that M_1, M_2, M_3 are positive semidefinite such that $M_1 + M_2 + M_3 = I_N$ and maximize the expected winning probability:

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- How small could it be?

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Thank you for your attention!