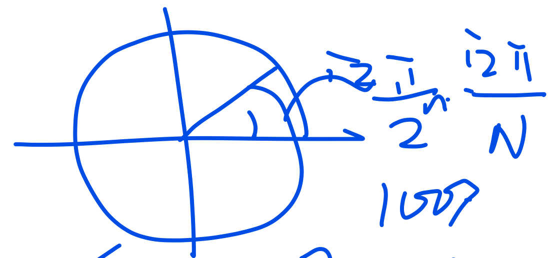
6.1 Quantum Integral Transform

Let $S_n = \{0, \ldots, N-1\}$ with $N = 2^n$ and let K be an $N \times N$ complex matrix with entries $K(i, j)$ with $i, j \in S_n$. Then K is a QIT transform converting $f = (f(0), \ldots, f(N-1))^t$ to

$$\tilde{f} = (\tilde{f}(0), \ldots, \tilde{f}(N-1))^t \qquad \text{by} \qquad \tilde{f} = Kf.$$

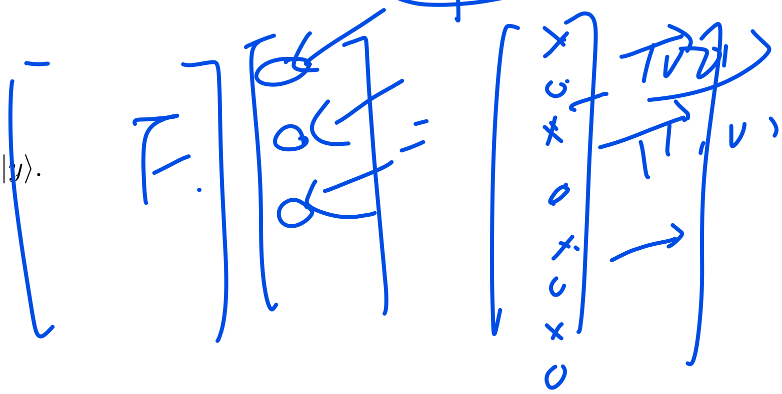


If K is unitary (invertible) then

$$f = K^\dagger \tilde{f} \qquad \text{(respectively, } f = K^{-1}\tilde{f}\text{)}.$$

Proposition If $U|x\rangle = K|y\rangle$, then

$$U\left[\sum_{x=0}^{2^n-1} f(x)|x\rangle\right] = \sum_{y=0}^{2^n-1} \tilde{f}(y)|y\rangle.$$



6.2 Quantum Fourier Transform

Suppose $N = 2^n$, $w = e^{2\pi i/N}/\sqrt{N}$ and

$K = K(x, y)$ with $K(x, y) = (w_n^{-xy})$.

Then $\tilde{f} = Kf$ is a commonly used QFT.

Example When $n = 1, 2$.

$$\underline{H \otimes H \otimes H |000\rangle}$$

$$= H \otimes H \otimes H \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \sum_{x=0}^{7} |x\rangle \quad \left\{\begin{matrix} |000\rangle \\ |001\rangle \\ \vdots \\ |111\rangle \end{matrix}\right.$$

U_f

$$\sum |x\rangle |f(x)\rangle$$

6.3 Application of QFT to period finding

This is an essential component in the Shor's algorithm.

For a periodic function, $f: S_n \to S_n$, where $S_n = \mathbb{Z}_2^n$, we want to detect $P \in S_n$ such that

$$f(x) = f(x+P) \quad \text{for all } x \in S_n.$$

Example Let $n = 3, P = 2$; $f(0) = f(2) = f(4) = f(6) = a$, $f(1) = f(3) = f(5) = f(7) = b$.

Step 1. Prepare $|\Psi_0\rangle = |0\rangle|0\rangle \in S_3 \otimes S_3$.
Step 2. Apply $W_3 \otimes I_8$ to $|\Psi_0\rangle$ and the oracle U_f to get $|\Psi\rangle = \gamma \sum_x |x\rangle|f(x)\rangle$.
Step 3. Apply $F = [e^{-2\pi ixy/8}] \otimes I_n$ to $|\Psi\rangle$ to get

$$\begin{aligned}
|\Psi'\rangle &= \gamma \sum_{x,y} e^{-2\pi ixy/8}|y, f(x)\rangle \\
&= \gamma|0\rangle[|f(0)\rangle + |f(1)\rangle + \cdots + |f(7)\rangle] \qquad (y=0) \\
&\quad +\gamma|1\rangle[|f(0)\rangle + e^{-2\pi i/8}|f(1)\rangle + \cdots + e^{-2\pi i7/8}|f(7)\rangle] \ (y=1) \\
&\quad + \qquad \cdots \quad \cdots \\
&\quad +\gamma|7\rangle[|f(0)\rangle + e^{-14\pi i/8}|f(1)\rangle + \cdots + e^{-14\pi i7/8}|f(7)\rangle] \ (y=7) \\
&= \frac{1}{2}(|0,a\rangle + |0,b\rangle + |4,a\rangle + e^{-i\pi}|4,b\rangle).
\end{aligned}$$

Step 4. Measurement of the first register gives $0, 4$. So the period is 2.

Remark Table 6.2 is not accurate.

TABLE 6.3
Coefficient of a vector $|y\rangle|f(x)\rangle$ in the state $|\Psi'\rangle$ in which $f(0) = f(2) = f(4) = f(6) = a$ and $f(1) = f(3) = f(5) = f(7) = b$. The amplitude of all the non-vanishing coefficients is $1/2$.

$\|b\rangle$	$\rightarrow$	0	0	0	$\leftarrow$	0	0	0
$\|a\rangle$	$\rightarrow$	0	0	0	$\rightarrow$	0	0	0
	$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$	$\|4\rangle$	$\|5\rangle$	$\|6\rangle$	$\|7\rangle$

Remark In general, the observed value of the first register is one of

$$\frac{1}{P}k \cdot 2^n, \qquad k = 0, 1, \ldots, P-1.$$

$\hat{W}\ U|\psi\rangle = |\psi\rangle$

6.4 Implementation of QFT

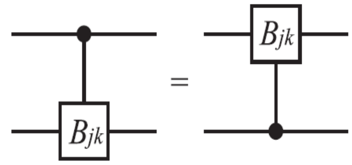
When $n = 1$. $U_{QFT_1} = W_1$.

When $n \geq 2$, we need the controlled

$$B_{jk} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i2\pi/(k-j+1)} \end{pmatrix} \text{ for } k \geq j$$

and the SWAP gate to implement U_{QFT_2}.

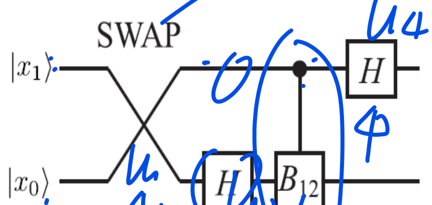
Note that



(a) (b)

PROPOSITION 6.3 The $n = 2$ QFT gate is implemented as

$$U_{\text{QFT2}} = (U_H \otimes I)U_{12}(I \otimes U_H)U_{\text{SWAP}}$$



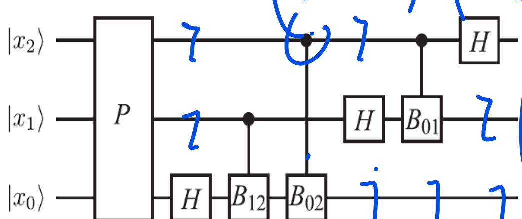
$$\frac{1}{2}\sum_{y=0}^{3} \omega_2^{-xy}|y\rangle$$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$I \otimes H$

Note $U_{QFT_n} = U^t_{QFT_n}$. So, ...

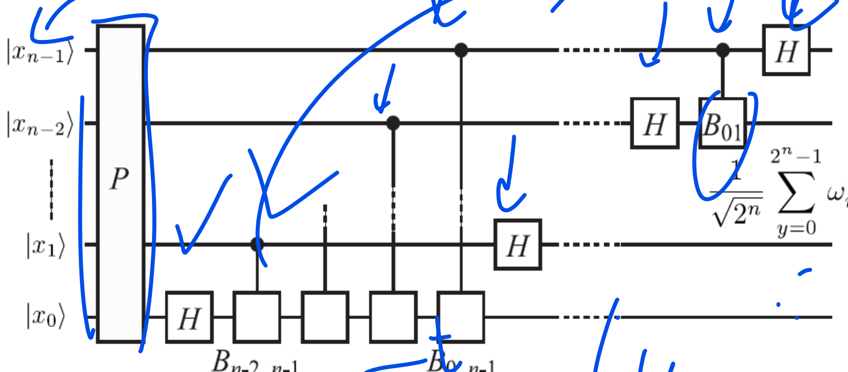
When $n = 3$, we have the following.



$$\frac{1}{\sqrt{2^3}}\sum_{y=0}^{7} \omega_3^{-xy}|y\rangle$$

$F_4 = F_{16}^t$

When $n \geq 3$, we have the following.



$F_n = F_n^t = (U_m \cdots U_1 P)^t = P^t U_1^t \cdots U_m^t$

Proposition QFT can be implemented using $O(n^2)$ elementary gates.

FUF

6.5 Walsh-Hadamard Transform

The kernel $W_n = ((-1)^{x\cdot y})$ defines the discrete integral transform

$$\tilde{f}(y) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{x\cdot y} f(x).$$

6.6 Selective Phase Rotation Transform

The kernel $\mathrm{diag}\,(\theta_0, \ldots, \theta_{N-1})$ defines the transform

$$\tilde{f}(y) = \sum e^{i\theta_x} \delta_{xy} f(x) = e^{i\theta_y} f(y).$$

Note that

$$K_1 = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix}, \quad K_2 = \begin{pmatrix} e^{i\theta_0} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix}.$$

Here

$$K_2 = A_0 A_1,$$

with

$$A_0 = \begin{pmatrix} e^{i\theta_0} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix},$$

$$A_0 = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes I, \quad U_0 = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix},$$

$$A_1 = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U_1, \quad U_1 = \begin{pmatrix} e^{i\theta_2} & 0 \\ 0 & e^{i\theta_3} \end{pmatrix}.$$

Remark One can actually write

$$K_2 = (I \otimes K_1)\hat{A}_2 \quad \text{with} \quad \hat{A}_1 = \mathrm{diag}\,(1, 1, e^{i(\theta_2-\theta_0)}, e^{i(\theta_2-\theta_1)}).$$