An invitation to Quantum Computing C.K. Li Week 8

Quantum computing/quantum algorithm

- Use quantum properties (superposition, measurement, etc.) to process/manipulate information.
- One needs to formulate the problem in terms of an n-qubit state (register) $|\psi_1\rangle$ in $S_n = \{0, \ldots, 2^{n-1}\}$.
- Very often, one has to use additional m-qubit state to get the register $|\psi_1\rangle|\psi_2\rangle$.
- Then apply quantum unitary operations $U_1, U_2, \ldots, U_k$ so that a measurement of the resulting state $|\psi\rangle$ will give you useful information with high probability.
- The challenge includes:

  * formulation of the problems and the use of other mathematical ideas such as continued fraction, group theory, etc.

  * designing efficient quantum operations and address practical issues in implementation.
- Researchers have connected the study to image processing, neural network, AI, etc.

Let us visit the IBM Q online textbook.

Quantum Information

- Information theory studies the transmission, processing, extraction, and utilization of information.
- Abstractly, information can be thought of as the resolution of uncertainty (from given data).
- Important topics include Entropy, Differential entropy, Conditional entropy, Joint entropy, Mutual information, Conditional mutual information, Relative entropy, Entropy rate, Limiting density of discrete points, Asymptotic equipartition property, Rate–distortion theory, Shannon's source coding theorem, Channel capacity, Noisy-channel coding theorem, Shannon–Hartley theorem (uncertainty of the given data), Error correction for noisy channels, etc.
- Quantum information science use quantum properties to study information theory.
- Some of these topics and backgrounds are mentioned in the supplementary notes in Week 3.

  https://cklixx.people.wm.edu/teaching/QC2021/QC-chapter3.pdf.
- Instead of telling you many different topics with my rather superficial understanding, let me share with you some of my current research topics on quantum tomography, quantum operations for open systems, and quantum error corrections.

Quantum State Tomography

- Recall that a measurement of $|\psi\rangle$ associated with a Hermitian matrix A will yield an eigenvalue of A and change "collapse" $|\psi\rangle$ to the corresponding eigenstate $|\phi_1\rangle, |\phi_2\rangle$ of A with probabilities $|\langle\phi_1|\psi\rangle|^2$ and $|\langle\phi_2|\psi\rangle|^2$.

- If one has many identical copies of $|\psi\rangle = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, one can estimate $|(1,0)|\psi\rangle|^2 = |u_1|^2$ and $|(0,1)|\psi\rangle|^2 = |u_2|^2$ by applying measurement associated with σ_z.

  One can also estimate $|u_1+u_2|^2/2$ and $|u_1-u_2|^2/2$ by applying measurement associated with σ_x.

  One can also estimate $|u_1-iu_2|^2/2$ and $|u_1+iu_2|^2/2$ by applying measurement associated with σ_y.

  Then we can estimate $|\psi\rangle$.

- In fact, if we consider the pure state

  $$|\psi\rangle\langle\psi| = \frac{1}{2}(I_2 + c_x\sigma_x + c_y\sigma_y + c_z\sigma_z) = \frac{1}{2}\begin{pmatrix} 1+c_z & c_x - ic_y \\ c_x + ic_y & 1-c_z \end{pmatrix},$$

  the above procedures give the estimates of c_z, c_x, c_y, respectively.

- The same procedures yield the estimate for general density matrix $\rho \in D_2$,

- The process of estimating a mixed state ρ by apply measurements to an ensemble of identical copies of states is called **quantum state tomography**.

Multi-qubit state tomography using local measurements

- For a 2-qubit state $\rho \in D_4$, one can estimate ρ using measurements operators in

$$\mathcal{S}_2 = \{A_1 \otimes A_2 : A_1, A_2 \in \{\sigma_x, \sigma_y, \sigma_z\}\}.$$

- This is because if two (Hermitian) matrix $X, Y \in \mathbf{M}_4$ give the same measurements for all matrices in $\mathcal{S}_2$, then $X - Y = aI$ for some $a \in \mathbf{C}$.

- It is interesting to note that one can determine/estimate any (entangled or separable or tensor) state ρ by doing local measurements on 9 measurement bases.

- One can extend the result to an n-qubit state ρ using 3^n measurement operators in

$$\mathcal{S}_2 = \{A_1 \otimes \cdots \otimes A_n : A_1, \ldots A_n \in \{\sigma_x, \sigma_y, \sigma_z\}\}.$$

- We have implemented the scheme using IBM Q computer. The measurement error is terrible!

- **Open question.**

  Can we use fewer than 3^n measurement bases if $n \geq 2$?

Use $2^n + 1$ measurement bases for n-qubit states

- Note that one needs $N^2 - 1$ real data to specific $\rho \in D_N$ with $N = 2^n$.
- Every measurement operator yield $N - 1$ piece of information.
- So, $N + 1$ measurement bases measurement bases are needed to estimate/determine $\rho \in D_N$.
- Of course, the measurement operators cannot be all local, else, we cannot differentiate

  $$\rho_1 = a_1 I_2 \otimes \cdots \otimes a_n I_2 \text{ and } \rho_2 = b_1 I_2 \otimes \cdots \otimes b_n I_2.$$
- We have some success in using IBM Q to carry out the scheme for 2-qubit states. But there are still much error.

Assisted tomography schemes

- Suppose an n-qubit state $\rho \in D_N$ is given with $N = 2^n$.
- We consider $\sigma \otimes \rho = \begin{pmatrix} \rho & 0 \\ 0 & 0_{N^2-N} \end{pmatrix} \in D_{N^2}$.
- We then apply a suitable measurement basis in $\mathbf{M}_{N^2}$, equivalently, choose a suitable unitary $U \in U(N^2)$ and estimate the $N^2 - 1$ diagonal entries of $U(\sigma \otimes \rho)U^\dagger$.
- We can then use these data to determine ρ.
- Mathematically, it means that $\rho \mapsto \operatorname{diag}(U(\sigma \otimes \rho)U^\dagger)$ is a (linear) bijection.

Density matrix best fits the measurement values

After getting the measurements of $U_j \rho U_j^\dagger$ for all j, or $U(\sigma \otimes \rho)U^\dagger$, one needs to find $\tilde{\rho} \in D_n$ which give the measurements or best fit the measurements.

One can use different methods to find $\tilde{\rho}$ such as

- maximum likelihood - use a suitable penalty function to search for $\tilde{\rho}$,
- linear inversion and best approximation - use least square method to solve for the Hermitian matrix $\hat{\rho}$ attaining the measurement values, and find the best density matrix $\tilde{\rho}$ approximating $\hat{\rho}$.
- projection method - use projection method to find the density matrix which is nearest to the linear manifold satisfying the measurement values.

Further research

- One may use different device to do the measurement.
- For example, using NMR, one will get the $(1,2),(1,3),(2,3),(2,4)$ entries of $\rho \in D_4$ in one experimental set up.

  (Setting up the interaction between the atoms in the molecules.)

  So, measuring ρ and $U\rho U^\dagger$ will be enough to estimate $\rho \in D_4$.
- For $\rho \in D_8$, one may get 8 or 12 upper triangular entries of ρ depending on linear interaction or complete interaction of the three quits. So, measuring ρ and $U\rho U^\dagger$ can determine ρ.
- Also, if $\sigma \in D_4$, then a measurement of $U(|0\rangle\langle 0| \otimes \sigma)U^\dagger$ will determine σ.
- We still have to find the quantum state $\tilde{\rho}$ which best fits the measurement values.
- One may consider other quantum device, say, linear optics.

Quantum process tomography

- It is also of interested to determine a given quantum process / operation.
- Suppose $\Phi : M_n \to M_m$. We can determine Φ by testing $\Phi(\rho_j)$ for a linearly independent set $\{\rho_1, \dots, \rho_{N^2-1}\}$ of states in M_n.
- It is known that $\Phi : M_n \to M_m$ if and only if $C(\Phi) = \frac{1}{m}(\Phi(E_{ij})$ is a quantum state ρ_Φ such that

  $$\mathrm{Tr}_1(\rho_\Phi) = \frac{1}{m}(\mathrm{Tr}\Phi(E_{ij})) = \frac{1}{m}I_m.$$

  We are trying to adapt the techniques in quantum state tomography for the study.
- A quantum operation $\Phi : M_n \to M_n$ of a closed system has the form $A \mapsto UAU^\dagger$. One have to determine U.

  In this case, ρ_Φ is a pure state, and there may be more efficient method to estimate ρ_Φ.