Quantum operations with special properties

Image processing

- A black and white picture can be stored as an $m \times n$ matrix $A = (a_{ij}) \in M_{m,n}$
- Each entry $a_{ij} \in [0, 1]$ is a "pixel" with a certain grey level.
- A color picture can be encoded store as three matrices $A_r, A_b, A_g \in M_{m,n}$ using the three baic colors: red, blue, green.
- One may process the image by manipulating the matrix A.
- Some examples:

  * Data compression: singular value decomposition.

  * Special pattern finding: find some special patterns, say, finding roads, edges, different geographical features.

  * Image recognition: comparing a given picture with some given pictures.

  * Classification (AI) problems: classify pictures with similarity.
- Question. How does quantum computer help?

Basic question

How to encode $A \in M_{m,n}$ as a quantum state $|v\rangle \in \mathbf{C}^{mn}$?

We can assume $m = 2^p$ and $n = 2^q$. Then we can assume $|v\rangle \in 2^{p+q}$, arranging the entries of first rows, second rows, etc.

Answer Need to find simple unitary U such that $V|0\cdots 0\rangle = |v\rangle$, equivalently, $U|v\rangle = |0\cdots 0\rangle$.

- For $|v\rangle \in \mathbf{C}^2$, we can use $U \in \mathbf{U}(2)$ such that $U|v\rangle = |0\rangle$.
- For $|v\rangle \in \mathbf{C}^4$, we can find $U_1, U_2, U_3 \in \mathbf{U}(2)$ such that

  $$(I_2 \otimes U_1)|v\rangle = (c, 0, s_1, s_2)^t,$$

  $$(I_2 \oplus U_2)(c, 0, s_1, s_2)^t = c, 0, s, 0)^t,$$

  $$(U_3 \otimes I_2)(c, 0, s, 0)^t = (1, 0, 0, 0)^t.$$

  So, we can use two 0-controlled, one 1-controlled qubit gates.
- For $|v\rangle \in \mathbf{C}^8$, we use $U_1, U_2 \in \mathbf{U}(4), U_0 \in \mathbf{U}(2)$ such that

  $$I_2 \otimes U_1|v\rangle = |v_1\rangle = (c, 0, 0, 0, s_1, s_2, s_3, s_4)^t,$$

  $$I_4 \otimes U_2|v_1\rangle = (c, 0, 0, 0, s, 0, 0, 0)^t,$$

  which can be change to $|000\rangle$ by a 0-controlled gate.
- Let $c_{n,k}$ be the number of k-controlled gates needed for $k = 0, \dots, n-1$. Then $c_{1,0} = 1, \quad (c_{2,0}, c_{2,1}) = (2, 1)$, $(c_{3,0}, c_{3,1}, c_{3,2}) = (2, 1, 0) + (0, 2, 1) + (1, 0, 0) = (3, 3, 1)$.
- In general, $c(n, k) = \binom{n}{k+1}$.

Quantum states with specific images

Question Given $\{|u_1\rangle, \dots, |u_k\rangle\}, \{|v_1\rangle, \dots, |v_k\rangle\} \subseteq \mathbf{C}^n$, does there exist $U \in \mathbf{U}(n)$ such that

$$U|u_j\rangle = |v_j\rangle, \qquad j = 1, \dots, k.$$

Answer The two Gram matrices $(\langle u_i|u_j\rangle), (\langle v_i|v_j\rangle) \in M_k$ are equal.

Question Given $\{|u_0\rangle, |u_1\rangle, \dots\} \subseteq \mathbf{C}^n$, does there exist $U \in \mathbf{U}(n)$ such that

$$U|u_j\rangle = |u_{j+1}\rangle, \quad j = 0, 1, 2, \dots.$$

Answer The matrix $(\langle u_i|u_j\rangle)_{i,j=0,1,\dots} = (\langle u_i|u_j\rangle)_{i,j=1,2,\dots}$, i.e., the matrix is Toeplitz.

Actually, we only need to check the leading $n \times n$ submatrix, or $k \times k$ submatrix if $\text{span}\{|u_j\rangle : j = 0, 1, \dots\}$ has dimension k.

Results for open systems

Recall that mixed states are density matrices in M_n. A general quantum operation $\Phi : M_n \to M_m$ is a TPCP maps admitting the operator sum representation

$$\Phi(A) = F_1 A F_1^\dagger + \cdots + F_r A F_r^\dagger \quad \text{for all} \quad A \in M_n$$

for some $m \times n$ matrices $F_1, \ldots, F_r$ satisfying $F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n$.

The following result is due to A. Chefles, R. Jozsa, and A. Winter, 2004.

Theorem Let $\{|u_1\rangle, \ldots, |u_k\rangle\} \subseteq \mathbf{C}^n$ and $\{|v_1\rangle, \ldots, |v_k\rangle\} \subseteq \mathbf{C}^m$. There is a quantum operation $\Phi : M_n \to M_m$ satisfying

$$\Phi(|u_j\rangle\langle u_j|) = |v_j\rangle\langle v_j| \qquad \text{for all } j = 1, \ldots, k,$$

if and only if there is a correlation matrix $C = (c_{ij})$ such that

$$(\langle u_i|v_j\rangle) = C \circ (\langle v_i|v_j\rangle),$$

the Schur product (a.k.a. Hadamard or entry-wise product), i.e.,

$$\langle u_i|u_j\rangle = c_{ij}\langle v_i|v_j\rangle \quad \text{for all } 1 \le i, j \le k.$$

Some general results

In 2012, Z. Huang, C.K. Li, E. Poon and N.S. Sze, obtained some general results for the existence of TPCP map $\Phi(A_j) = B_j$ for $j = 1, \ldots, k$, with $\{A_1, \ldots, A_k\} \subseteq D_n, \{B_1, \ldots, B_k\} \subseteq D_m$. There were results for diagonal matrices [Li and Y. Poon, 2011], and compact diagonal operators [Hsu, Kuo, Tsai, 2014].

Theorem Let

$$\{A_j = |u_j\rangle\langle u_j| : 1 \leq j \leq k\} \subseteq D_n \text{ and } \{B_1, \ldots, B_k\} \subseteq D_m.$$

There is $\Phi : M_n \to M_m$ such that $T(A_j) = B_j$ for $j = 1, \ldots, k$ if and only if there is a purifcation of $|v_j\rangle\langle v_j|$ of B_j for $j = 1, \ldots, k$ such that $(\langle u_i|u_j\rangle) = (\langle v_i|v_j\rangle)$.

- The general condition for Φ sending mixed states to mixed states are very technical.
- It depends on the spectral decomposition, solution of certain matrix equations, etc.

More results and questions

- For any $\rho \in D_n, \sigma \in D_m$, the map $A \mapsto (\text{Tr}A)\sigma$ is a TPCP map sending all states to σ.
- Let $A_1, A_2 \in D_n, B_1, B_2 \in D_m$. The condition of the existence of a TPCP map $\Phi : M_n \to M_m$ such that

  $$(\Phi(A_1), \Phi(A_2)) = (B_1, B_2), \text{ i.e., } \Phi(A_1 + iA_2) = B_1 + iB_2$$

  is not known.
- For qubit states, we may assume that A_1, A_2 are pure state. Then Φ exists if and only if

  $$\text{Tr}\sqrt{A_1^{1/2} A_2 A_1^{1/2}} \leq \text{Tr}\sqrt{B_1^{1/2} B_2 B_1^{1/2}}.$$
- Suppose $\{A_1, \ldots, A_4\} \subseteq D_2$ are linearly independent. There is a unique linear map satisfying $\Phi(A_j) = B_j$ for $j = 1, \ldots, 4$. It is then easy to determine whether Φ is TPCP.

- Suppose $\{A_1, A_2, A_3\}, \{B_1, B_2, B_3\} \subseteq D_2$ such that

  $A_j = |u_j\rangle\langle u_j|$ for $j = 1, 2, 3$, are linearly independent.

  Let $|u_3\rangle = \alpha_1|u_1\rangle + \alpha_2|u_2\rangle$, and $\hat{B}_3 = |\alpha_1 u_1\rangle\langle\alpha_2 u_2| + |\alpha_2 u_2\rangle\langle\alpha_1 u_1|$. Then there is a TPCP map sending A_j to B_j for $j = 1, 2, 3$, if and only if there is $C \in M_2$ such that

  $$\mathrm{Tr}(CC^*) = 1 + |\det(C)|^2 \le 2, \ \hat{B}_3 = \mathrm{Re}(\sqrt{B_2}C\sqrt{B_1}),$$

  $$\mathrm{Tr}\sqrt{B_2}C\sqrt{B_1} = \langle\alpha_1 u_1|\alpha_2 u_2\rangle.$$

- Question. Find a simpler condition.

- Current research with Ray-Kuang Lee. Let $\{\rho_0, \rho_1, \dots\} \subseteq D_n$. Determine TPCP maps Φ such that $\Phi(\rho_j) = \rho_{j+1}$ for $j = 0, 1, \dots$.