1. Write each of the following sets as specified.

   (a) List the elements in the set \( A = \{ n \in \mathbb{N} : n^3 < 1000 \} \).
   
   Answer: \( A = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \} \).

   (b) Describe the set \( B = \{ -2, -1, 0, 1, 2, 3, 4 \} \) using the notation \( \{ n : p(n) \} \), where \( p(n) \) specifies the property of element \( n \).
   
   Answer: \( B = \{ n \in \mathbb{Z} : n < -3 \} \)

2. Recall that for a set \( A \), \( \mathcal{P}(A) \) denotes the power set of \( A \).

   (a) Find \( \mathcal{P}(\{a\}) \) and its cardinality.
   
   Solution: \( \mathcal{P}(\{a\}) = \{ \emptyset, \{a\} \} \).
   
   (b) Give an example of a set \( S \) such that \( S \subseteq \mathcal{P}(\mathbb{N}) \) and \( |S| = 5 \).
   
   Solution: \( S = \{ 1, 2, 3, 4, 5 \} \).

3. The following problems involve set operations.

   (a) Give an example of three two element sets \( A, B \), and \( C \) such that \( B \neq C \) but \( B - A = C - A \).
   
   Solution: Let \( A = \{ 1, 2 \}, B = \{ 1, 3 \} \) and \( C = \{ 2, 3 \} \). Then \( B - A = C - A = \{ 3 \} \).

   (b) Let \( A = \{ \emptyset, \{ \emptyset \}, \{ \{ \emptyset \} \} \} \). Find \( \mathcal{P}(A) - A \).
   
   Solution: \( \mathcal{P}(A) - A = \{ \{ \emptyset \}, A \} \).

4. For a real number \( r \), define \( S_r \) to be the interval \([ r - 1, r + 2 \). Let \( A = \{ 1, 3, 4 \} \).

   Determine \( \cup_{\alpha \in A} S_{\alpha} \) and \( \cap_{\alpha \in A} S_{\alpha} \).

   (a) List the intervals \( S_r \) for \( r \in A \).
   
   Solution. \( S_1 = [0, 3], S_3 = [2, 5], S_4 = [3, 6] \).

   (b) Determine \( \cup_{\alpha \in A} S_{\alpha} \) and \( \cap_{\alpha \in A} S_{\alpha} \).
   
   Solution. \( \cup_{\alpha \in A} S_{\alpha} = S_1 \cup S_3 \cup S_4 = [0, 6]; \quad \cap_{\alpha \in A} S_{\alpha} = S_1 \cap S_3 \cap S_4 = [2, 3] \cap [3, 6] = \emptyset \).

   (c) (Extra 3 points) Let \( B = [0, 1] \). Determine \( \cap_{r \in B} S_r \) and \( \cup_{r \in B} S_r \). (Give explanation.)
   
   Solution. We have \( \cap_{r \in B} S_r = [0, 2] \). To prove “\( \subseteq \)” if \( x \in \cap_{r \in B} S_r \), then \( x \in S_r = [r - 1, r + 2] \) for every \( r \in [0, 1] \). So, \( x \in S_0 = [-1, 2] \) and \( x \in S_1 = [0, 3] \), and hence \( x \in [0, 2] \).

   To prove “\( \supseteq \)” if \( x \in [0, 2] \), then \( r - 1 \leq x < r + 2 \) for every \( r \in [0, 1] \). So, \( x \in \cap_{r \in B} S_r \).

   We have \( \cup_{r \in B} S_r = [-1, 3] \). To prove “\( \subseteq \)” if \( x \in \cup_{r \in B} S_r \), then \( x \in S_r = [r - 1, r + 2] \) for some \( r \in [0, 1] \). So, \( -1 = 0 - 1 \leq r - 1 \leq x < r + 2 < 1 + 2 = 3 \), i.e., \( x \in [-1, 3] \).

   To prove “\( \supseteq \)” if \( x \in [-1, 3] \), then \( x \in S_0 \cup S_1 \), i.e., \( x \in S_0 \) or \( x \in S_1 \). Thus, \( x \in \cup_{r \in B} S_r \).
5. For two sets $A$ and $B$, recall that $A \times B$ is the Cartesian product of $A$ and $B$.

(a) Let $A = \{a, b\}$. Determine $A \times \mathcal{P}(A)$.

Solution. $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So, $A \times \mathcal{P}(A) = \{(a, \emptyset), (a, \{a\}), (a, \{b\}), (a, \{a, b\}), (b, \emptyset), (b, \{a\}), (b, \{b\}), (b, \{a, b\})\}.$

(b) Let $A = \{0, 1\}$ and $B = [0, 2] \cap [1, 3]$. Depict or describe geometrically the set $A \times B$ in $\mathbb{R}^2$.

Solution. $A \times B = L_1 \cup L_2$ where $L_1 = \{(0, b) : 1 \leq b \leq 2\}$ is the line on the $x - y$ plane joining the points $(0, 1)$ and $(0, 2)$, and $L_2 = \{(1, b) : 1 \leq b \leq 2\}$ is the line on the $x - y$ plane joining the points $(1, 1)$ and $(2, 2)$.

(c) Let $A = \{0, 1\}, B = (0, 1) \cap A$ and $C = \mathbb{R}$. What is $A \times B \times C$?

Solution. Note that $B = \emptyset$. So, there is no triple of the form $(a, b, c)$ with $a \in A, b \in B, c \in C$. Thus, $A \times B \times C = \emptyset$.

6. Determine all different partitions of the set $\{1, 2, 3\}$.

Solution. $P_1 = \{\{1\}, \{2\}, \{3\}\}; P_2 = \{\{1, 2\}, \{3\}\}; P_3 = \{\{1\}, \{2, 3\}\}; P_4 = \{\{1, 3\}, \{2\}\}; P_5 = \{\{1, 2, 3\}\}$.

7. Extra credit problem. If a set $A$ has $n$ elements, show that $\mathcal{P}(A)$ has $2^n$ elements.

Suppose $A = \{a_1, \ldots, a_n\}$. To form a subset, we have to decide whether the subset contains or not contain $a_k$ for $k = 1, \ldots, n$. For each $a_k$ there are two possible answer: Yes or No. So, there are $2^n$ possible different answers for the $n$ questions. In particular, if we say no to each $a_k$, then we get the empty set; if we say yes to each $a_k$, then we get the whole set $A$. Thus, the $2^n$ different answers for the $n$ questions will give rise to a different subsets. So, there are $2^n$ subsets.