1. Write each of the following sets as specified.
   (a) List the elements in the set \( A = \{ n \in \mathbb{N} : n^3 < 1000 \} \).
   (b) Describe the set \( B = \{-2, -1, 0, 1, 2, 3, 4\} \) using the notation \( \{ n : p(n) \} \), where \( p(n) \) specifies the property of element \( n \).
   Answer: (a) \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \).
   (b) \( B = \{ n \in \mathbb{Z} : -3 < n < 5 \} \).

2. Recall that for a set \( A \), \( P(A) \) denotes the power set of \( A \).
   (a) Find \( P(P(\{a\})) \) and its cardinality.
   (b) Give an example of a set \( S \) such that \( S \in P(\mathbb{N}) \) and \( |S| = 5 \).
   (c) Give an example of a set \( S \) such that \( S \subseteq P(\mathbb{N}) \) and \( |S| = 5 \).
   Answer: (a) \( P(\{a\}) = \{\emptyset, \{a\}\}; P(P(\{a\})) = \{\emptyset, \emptyset, \{\{a\}\}, \{\emptyset, \{a\}\}\} \) has cardinality 4.
   (b) \( S = \{1, 2, 3, 4, 5\} \). [Clearly, \( S \subset \mathbb{N} \). So, \( S \in P(\mathbb{N}) \).]
   (c) \( S = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\} \). [Clearly, every \( x \in S \) is an element of \( P(\mathbb{N}) \). So, \( S \subseteq P(\mathbb{N}) \).]

3. The following problems involve set operations.
   (a) Give an example of three two-element sets \( A, B, \) and \( C \) such that \( B \neq C \) but \( B - A = C - A \).
   (b) Let \( A = \{\emptyset, \{\emptyset\}\} \). Find \( P(A) - A \).
   Answer: (a) Let \( A = \{2, 3\} \), \( B = \{1, 2\} \), and \( C = \{1, 3\} \). Then \( B - A = C - A = \{1\} \).
   (b) \( P(A) - A = \{\emptyset, \{\emptyset\}, \{\emptyset\}, A\} - A = \{\{\emptyset\}, A\} \).

4. For a real number \( r \), define \( S_r \) to be the interval \( [r - 1, r + 2] \). Let \( A = \{1, 3, 4\} \). Determine \( \cup_{\alpha \in A} S_\alpha \) and \( \cap_{\alpha \in A} S_\alpha \).
   (a) List the intervals \( S_r \) for \( r \in A \).
   (b) Determine \( \cup_{\alpha \in A} S_\alpha \) and \( \cap_{\alpha \in A} S_\alpha \).
   (c) (Extra 3 points) Let \( B = [0, 1] \). Determine \( \cap_{r \in B} S_r \) and \( \cup_{r \in B} S_r \).
   Solution. (a) \( S_1 = [0, 3], S_3 = [2, 5], S_4 = [3, 6] \).
   (b) \( \cup_{\alpha \in A} S_\alpha = S_1 \cup S_3 \cup S_4 = [0, 6] \); \( \cap_{\alpha \in A} S_\alpha = S_1 \cap S_3 \cap S_4 = \emptyset \).
   (c) Let \( B = [0, 1] \). Determine \( \cap_{r \in B} S_\alpha \) and \( \cup_{\alpha \in (0,1)} S_\alpha \).
   Let \( X = \cap_{r \in B} S_r \). First, we show that \( [0, 2] \subseteq X \). If \( x \in [0, 2] \), then \( r - 1 \leq 0 \leq x < 2 \leq r + 2 \) for every \( r \in [0, 1] \). So, \( [0, 2] \subseteq X \). Next, we show that \( X \subseteq [0, 2] \). If \( x \in X \), then \( x \in S_r = [r - 1, r + 2] \) for every \( r \in [0, 1] \). In particular, \( x \in S_0 \) and \( x \in S_1 \). So, \( 0 \leq x < 2 \).
   Next, let \( Y = \cup_{r \in B} S_r \). We show that \( B = [-1, 3] \). Note that \( S_r = [r - 1, r + 2] \subseteq [-1, 3] \) for every \( r \in [0, 1] \). Thus, \( B \subseteq [-1, 3] \). Now, we show that \( [-1, 3] \subseteq B \). Suppose \( x \in B \). Then \( x \in [r - 1, r + 2] \) for some \( r \in [0, 1] \). So, \( -1 \leq r - 1 \leq x < r + 2 = 3 \). So, \( x \in [-1, 3] \).
5. For two sets $A$ and $B$, recall that $A \times B$ is the Cartesian product of $A$ and $B$.

(a) Let $A = \{a, b\}$. Determine $A \times \mathcal{P}(A)$.

(b) Let $A = \{0, 1\}$ and $B = [0, 2] \cap [1, 3]$. Depict or describe geometrically the set $A \times B$ in $\mathbb{R}^2$.

(c) Let $A = \{0, 1\}$, $B = (0, 1) \cap A$ and $C = \mathbb{R}$. What is $A \times B \times C$?

Answer. (a) $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So,

$$A \times \mathcal{P}(A) = \{(a, \emptyset), (a, \{a\}), (a, \{b\}), (a, \{a, b\}), (b, \emptyset), (b, \{a\}), (b, \{b\}), (b, \{a, b\})\}.$$

(b) $A \times B = L_1 \cup L_2$, where $L_1 = \{(0, b) : 1 \leq b \leq 2\}$ and $L_2 = \{(1, b) : 1 \leq b \leq 2\}$.

Geometrically, $L_1$ is the line on the $x - y$ plane joining the points $(0, 1)$ and $(0, 2)$, and $L_2$ is the line on the $x - y$ plane joining the points $(1, 1)$ and $(1, 2)$, and all the endpoints of the line segments are included.

(c) Note that $B = \emptyset$. So, there is no triple of the form $(a, b, c)$ with $a \in A, b \in B, c \in C$. Thus, $A \times B \times C = \emptyset$.

6. Determine all different partitions of the set $\{1, 2, 3\}$.

Answer. The set of partitions of $\{1, 2, 3\}$ equals $\{P_1, \ldots, P_5\}$, where $P_1 = \{\{1\}, \{2\}, \{3\}\}$; $P_2 = \{\{1, 2\}, \{3\}\}$; $P_3 = \{\{1\}, \{2, 3\}\}$; $P_4 = \{\{1, 3\}, \{2\}\}$; $P_5 = \{\{1, 2, 3\}\}$.

Extra credit problem. If a set $A$ has $n$ elements, show that $\mathcal{P}(A)$ has $2^n$ elements.

Solution. Suppose $A = \{a_1, \ldots, a_n\}$ has cardinality $n$. To form a subset, we have to decide whether the subset contains or not contain $a_k$ for $k = 1, \ldots, n$. For each $a_k$ there are two possible answers: Yes or No. So, there are $2^n$ possible different answers for the $n$ questions. In particular, if we say no to each $a_k$, then we get the empty set; if we say yes to each $a_k$, then we get the whole set $A$. Thus, each of the $2^n$ different answers for the $n$ questions will give rise to a different subset. So, $A$ has $2^n$ subsets.