1. (4 points) Let \( P: 25 \text{ is odd}, \) \( Q: 15 \text{ is prime} \) and \( R: \frac{1}{3} \in \mathbb{N} \). State each of the following in words, and determine whether they are true or false.

(a) \( P \lor Q \)
(b) \( P \land Q \)
(c) \( \sim P \lor Q \)
(d) \( P \land (\sim Q \lor R) \)

2. (4 points) In each of the following, two open sentences \( P(x) \) and \( Q(x) \) over a domain \( S \) are given. For each part, determine \( T = \{ x \in S : "P(x) \Rightarrow Q(x)" \text{ is true} \} \) with explanation.

(a) \( P(x) : x - 3 = 5; Q(x) : x > 8; S = \mathbb{N}. \)
(b) \( P(x) : x \in [-1, 3); Q(x) : x^2 \leq 4; S = [-10, 10]. \)

3. (4 points) Let \( P, Q \) be statements. Show that \( \sim (P \implies Q) \) and \( P \land (\sim Q) \) are logically equivalent using truth table.

Solution.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \implies Q )</th>
<th>( \sim (P \implies Q) )</th>
<th>( P \land (\sim Q) )</th>
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4. (6 points) Write the statements so that there are no \( \sim \) symbols. Then, rewrite the statements so that there are no \( \forall, \exists, \in \) or \( = \) symbols.

(a) \( \sim (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1) \);
(b) \( \sim (\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy = 0) \);
(c) \( \sim (\exists n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m \leq n) \);

5. (4 points) Consider the statement:
   “For every integer \( n > 0 \) there is some real number \( x > 0 \) such that \( x < 1/n. \)”
(a) Without using words of negation, write a complete sentence that negates the sentence.
(b) Determine the original statement or its negation is true with explanation.

6. (8 points) For \( \alpha > 0 \), let \( S_\alpha = (-\alpha, \alpha) \), i.e., the open interval with endpoints \(-\alpha, \alpha\). Prove or disprove the following statements. (Note that “\( \subset \)” means proper subset.)

(a) \( \forall \alpha \in (0, 1), \exists \beta \in (0, 1), S_\alpha \subset S_\beta. \)
    [Hint: Find \( \beta \) if \( \alpha = 0.9, 0.99, 0.999 \), etc. and find the general rule for specifying \( \beta \) for a given \( \alpha \).]
(b) \( \exists \alpha \in (0, 1), \forall \beta \in (0, 1), S_\alpha \subset S_\beta. \)
    [Hint: For \( \alpha = 0.1, 0.01, 0.001 \), etc. see whether one has \( S_\alpha \subset S_\beta \) for all \( \beta \in (0, 1) \). Then ...]