1. (7 points) Let $A, B, C$ be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

Hint: You may use any one of the following three approaches.

a) Write $(A - B) \cup (A - C) = \{ x \in U : p(x) \}$, where $p(x) : x \in (A - B)$ or $x \in (A - C)$, and $A - (B \cap C) = \{ x \in U : q(x) \}$, where $q(x) : x \in A$ and $x \notin (B \cap C)$, where $U$ is the universal set. Show that $p(x)$ and $q(x)$ are logically equivalent.

b) Show that if $x \in (A - B) \cup (A - C)$, then $x \in A - (B \cap C)$. Also, show that if $x \in A - (B \cap C)$, then $x \in (A - B) \cup (A - C)$.

c) Use set operations such as $A - X \subseteq A - Y$ if $Y \subseteq X$, to argue $(A - B) \cup (A - C) \subseteq A - (B \cap C)$ and also $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.

2. (7 points) Let $A, B, C$ and $D$ be sets. Prove that

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

Hint: Show that (a) if $(x, y) \in (A \times B) \cap (C \times D)$, then $(x, y) \in (A \cap C) \times (B \cap D)$, and (b) if $(x, y) \in (A \cap C) \times (B \cap D)$, then $(x, y) \in (A \times B) \cap (C \times D)$.

3. (7 points) For the following, state whether they are true or not. Then, prove your answer.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1$;
(b) $\exists n \in \mathbb{N}, \exists m \in (N - \{1\}), nm = 1$.

Hint: If you want to prove that $P$ is FALSE, you may try to prove $\sim P$ is TRUE.

4. (7 points) Show that for any two positive numbers $a$ and $b$,

$$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4.$$

Hint: Reduce the problem to $(a + b)^2 \geq 4ab$, and use algebra.

5. (7 points) Let $m = 4s + 2$ with $s \in \mathbb{Z}$. Show that there are no integers $x, y$ such that

$$x^2 - y^2 = m.$$

6. (7 points) Prove that the product of an irrational number and a nonzero rational number is irrational.

Hint: Assume that $x$ is irrational and $y$ is nonzero rational. If $xy$ is rational, then ...

7. (8 points) Let $S = \{a, b, c\} \subseteq \mathbb{Z}$. For any non-empty subset $X$ of $S$, let $s(X)$ be the sum of elements in $X$. Show that there are non-empty subsets $A, B$ of $S$ such that $s(A) = s(B)$ is divisible by 6.

Hint: For each non-empty subset $X$ of $S$, consider the remainder of $s(X)$ divided by 6. We get remainders $r_1, \ldots, r_7$. Show that two of these numbers are the same and deduce the result.

8. (Extra Credit, 8 points) Recall that for a given $S \subseteq \mathbb{R}$, the maximum element of $S$, denoted by $\max S$, is the number $\alpha \in S$ such that $\alpha \geq \beta$ for all $\beta \in S$.

Let $A = \{n \in \mathbb{N} \mid \sqrt{n} \notin \mathbb{Q}\}$. Show that $\max A$ does not exist.

Hint: Proof by contradiction. Suppose $N \in A$ is a maximum. Show that $n = 2N^2 \in A$ satisfy $n > N$. 

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