Five points for each question unless specified otherwise.

1. Let \( f_1 : A_1 \to B_1 \) and \( f_2 : A_2 \to B_2 \) be functions. Show that \( f : A_1 \times A_2 \to B_1 \times B_2 \) defined by \( f(a_1, a_2) = (f_1(a_1), f_2(a_2)) \) for every pair \((a_1, a_2) \in A_1 \times A_2\), is a function.
   
   (a) Show that if \( f_1, f_2 \) are injective. Then \( f \) is injective.
   
   (b) Show that if \( f_1, f_2 \) are surjective. Then \( f \) is surjective.

2. (a) Let \( f : A \to B \) be a function. Define a relation \( R \) on \( A \) by \( (a_1, a_2) \in R \) if \( f(a_1) = f(a_2) \). Show that \( R \) is an equivalence relation.
   
   (b) Suppose \( S \subseteq A \times B \). Define a relation \( \hat{S} \) on \( A \) by \( (a_1, a_2) \in \hat{S} \) if there is \( b \in B \) such that \( (a_1, b), (a_2, b) \in S \). Determine whether \( \hat{S} \) is reflexive, symmetric, transitive.

3. Construct \( f : A \to B \) and \( g : B \to A \) such that \( g \circ f = i_A \) and \( f \circ g \neq i_B \), where \( i_A \) is the identity function on \( A \) and \( i_B \) is the identity function on \( B \) in each of the following cases.
   
   (a) \( A = \{1\}, B = \{1, 2\} \).
   
   (b) \( A = B = \mathbb{N} \).

4. Let \( \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \) and \( \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix} \) be permutations in \( S_5 \).
   
   Determine \( \alpha \circ \beta, \beta \circ \alpha, \) and \( \beta^{-1} \).

5. Let \( A = \{2^a3^b : a, b \in \mathbb{N}\} \). Construct a bijection from \( \mathbb{N} \times \mathbb{N} \) to \( A \).
   
   In such a case, we say that \( \mathbb{N} \times \mathbb{N} \) and \( A \) has the same cardinality, denoted by \( |\mathbb{N} \times \mathbb{N}| = |A| \).

6. Show that \( f : \mathbb{R} \to (-1, 1) \) defined by \( f(x) = \frac{x}{1+|x|} \) is a bijection, i.e., \( |(-1, 1)| = |\mathbb{R}| \).

7. (a) Construct (with proof) a bijection from \( f : \{0\} \cup \mathbb{N} \to \mathbb{N} \).
   
   (b) Construct (with proof) a bijection from \( g : \mathbb{Q} \to \mathbb{Q} - \{0\} \).
   
   Hint for (b). Partition the domain into \( A_1 \cup A_2 \) with \( A_1 = \{0, 1, 2, \ldots\} \), and partition the co-domain into \( \mathbb{N} \cup A_2 \). Construct \( f : \mathbb{Q} \to \mathbb{Q} - \{0\} \) by \( f(x) = \begin{cases} x + 1 & \text{if } x \notin A_1, \\ x & \text{if } x \in A_2. \end{cases} \)

8. Show that \( f : (0, 1] \to (0, 1) \) defined by \[ f(x) = \begin{cases} \frac{1}{x+1} & \text{if } x = \frac{1}{n}, \\ \frac{1}{x} & \text{otherwise,} \end{cases} \]
   
   is a bijection.

9. (Extra credits) Construct a bijection \( f : \mathbb{Q} \times \mathbb{Z}_{16} \to \mathbb{Z} \times \mathbb{Z}_3 \), and describe \( f^{-1} \).