Six points for each question unless specified otherwise.

1. Let $f_1 : A_1 \rightarrow B_1$ and $f_2 : A_2 \rightarrow B_2$ be functions. Show that $f : A_1 \times A_2 \rightarrow B_1 \times B_2$ defined by $f(a_1, a_2) = (f_1(a_1), f_2(a_2))$ for every pair $(a_1, a_2) \in A_1 \times A_2$, is a function.

(a) Show that if $f_1, f_2$ are injective. Then $f$ is injective.

(b) Show that if $f_1, f_2$ are surjective. Then $f$ is surjective.

2. (a) Let $f : A \rightarrow B$ be a function. Define a relation $R$ on $A$ by $(a_1, a_2) \in R$ if $f(a_1) = f(a_2)$.

Show that $R$ is an equivalence relation.

(b) Suppose $S \subseteq A \times B$. Define a relation $\hat{S}$ on $A$ by $(a_1, a_2) \in \hat{S}$ if there is $b \in B$ such that $(a_1, b), (a_2, b) \in S$. Determine whether $\hat{S}$ is reflexive, symmetric, transitive.

3. Construct $f : A \rightarrow B$ and $g : B \rightarrow A$ such that $g \circ f = i_A$ and $f \circ g \neq i_B$, where $i_A$ is the identity function on $A$ and $i_B$ is the identity function on $B$ in each of the following cases.

(a) $A = \{1\}, B = \{1, 2\}$.

(b) $A = B = \mathbb{N}$.

4. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$ be permutations in $S_5$.

Determine $\alpha \circ \beta$, $\beta \circ \alpha$, and $\beta^{-1}$.

5. Let $A = \{2^a3^b : a, b \in \mathbb{N}\}$. Construct a bijection from $\mathbb{N} \times \mathbb{N}$ to $A$.

In such a case, we say that $\mathbb{N} \times \mathbb{N}$ and $A$ has the same cardinality, denoted by $|\mathbb{N} \times \mathbb{N}| = |A|$.

6. Show that $f : \mathbb{R} \rightarrow (-1, 1)$ defined by $f(x) = \frac{x}{1 + |x|}$ is a bijection, i.e., $|(-1, 1)| = |\mathbb{R}|$.

7. (a) Construct (with proof) a bijection from $f : \{0\} \cup \mathbb{N} \rightarrow \mathbb{N}$.

(b) Construct (with proof) a bijection from $g : \mathbb{Q} \rightarrow \mathbb{Q} - \{0\}$.

Hint for (b). Partition the domain into $A_1 \cup A_2$ with $A_1 = \{0, 1, 2, \ldots\}$, and partition the co-domain into $\mathbb{N} \cup A_2$. Construct $f : \mathbb{Q} \rightarrow \mathbb{Q} - \{0\}$ by $f(x) = \begin{cases} x + 1 & \text{if } x \in A_1, \\ x & \text{if } x \in A_2. \end{cases}$

8. Show that $f : (0, 1) \rightarrow (0, 1)$ defined by

$$ f(x) = \begin{cases} \frac{1}{x+1} & \text{if } x = \frac{1}{n}, \\ x & \text{otherwise}, \end{cases} $$

is a bijection.

9. (Extra credits) Construct a bijection $f : \mathbb{Q} \times \mathbb{Z}_{16} \rightarrow \mathbb{Z} \times \mathbb{Z}_3$, and describe $f^{-1}$. 
