Math 214 Homework 10

Your name

Four points for each problem unless specified otherwise.

1. Show that if $A$ and $B$ are denumerable sets, then $A \cup B$ is also denumerable
   
   [Hint: Let $A = \{a_1, a_2, \ldots\}$, and let $C = B - A$. Then $A \cup B = A \cup C$. Consider three cases: $C = \emptyset$, $C$ is finite, $C$ is denumerable.]

2. Prove that $S = \{(a, b) : a, b \in \mathbb{N}, a \geq 2b\}$ is denumerable.
   
   [Hint: Show that $S \subseteq \mathbb{N} \times \mathbb{N}$, which is denumerable, and that $S$ is not finite. Alternatively, show that $f : S \to \mathbb{N}$ defined by $f(a, b) = 2^a 3^b$ is a one-one function, and deduce that $|S| = |f(S)|$ is denumerable. You need to argue that $f(S)$ is not finite.]

3. For $k \in \mathbb{N}$, let $S_k = \{A \subseteq \mathbb{N} : |A| = k\}$. Show that $|S_2| = |\mathbb{N}|$
   
   [Hint: Show that $|S_2| = |\tilde{S}_2|$ where $\tilde{S}_2 = \{(a, b) : a, b \in \mathbb{N}, a < b\}$ and that $\tilde{S}_2 \subseteq \mathbb{N} \times \mathbb{N}$ is denumerable.]

4. (Extra 4 points) Using the definition of $S_k$ from problem 3, show that

   (a) for all $k \in \mathbb{N}$, $S_k$ is denumerable.
   (b) $S = \bigcup_{k=1}^{\infty} S_k$ is denumerable.

   [Hint: For (a), show that $|S_k| = |\tilde{S}_k|$, where $\tilde{S}_k = \{(a_1, dots, a_k) : a_1, \ldots, a_k \in \mathbb{N}, a_1 < \cdots < a_k\}$ is denumerable.
   For (b) Use the result below.]

5. Let $\emptyset \neq J \subseteq \mathbb{N}$. For each $j \in J$, $A_j$ is a non-empty countable set. Show that $\bigcup_{j \in J} A_j$ is countable.

   [Hint: Let $J = \{j_1, j_2, \ldots\}$ that may be finite or denumerable. Let $A_{j_\ell} = \{a_{\ell 1}, a_{\ell 2}, \ldots\}$ be nonempty countable (that may be finite1 or infinite) . Consider the list of primes $p_1, p_2, \ldots$ arranged in ascending order. Define $f : \bigcup_{j \in J} A_j \to \mathbb{N}$ by $f(a_{j,k}) = p_1^{a_{\ell 1}} p_2^{a_{\ell 2}} \cdots$ if $a_{j,k} \notin \bigcup_{\ell \in J} A_{j_{\ell}}$. Show that $f$ is injective and hence $|\bigcup_{j \in J} A_j| = |B|$ with $B = \{f(x) : x \in \bigcup_{j \in J} A_j\}$. Then derive the conclusion.]

6. Let $A = \{(a_1, a_2, a_3, \ldots) : a_i \in \{0, 1\}, i \in \mathbb{N}\}$. Show that there is no surjection $f : \mathbb{N} \to A$.

7. (6 points) Determine the cardinality of the following sets (finite, denumerable, or uncountable), and justify your answers:

   (a) the set of all open intervals with rational midpoints.
   (b) the set of all open intervals with rational endpoints.

8. Show that $f : (0, 1) \times (0, 1) \to (0, 1)$ defined by

   $$f(0.a_1a_2\cdots, 0.b_1b_2\cdots) = 0.a_1b_1a_2b_2a_3b_3\cdots$$

   is bijective.

9. Let $A = (0, 1) \cup (2, 3)$ and $B = [1, 2]$. Construct an injection from $A$ to $B$, and construct an injection from $B$ to $A$. 

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