Chapter 0 Communicating Mathematics

Read Chapter 0 carefully.

Remarks

- In learning, using, teaching, and research of mathematics, it is important to communicate the ideas with other people.

- We need to explain the concepts, ideas, methods, and reasoning (proof) of mathematical formulas, algorithms, statements (theorems, propositions, lemmas, corollaries), clearly.

- We will use mathematics symbols, and precise definitions of mathematical terms.

- You must remember the meaning of the symbols and definitions precisely.

- Basic number systems: \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C} \), and operations.

- Equations: \( x^2 - 6x + 8 = 0 \), \( \{x + y = 2, x - y = 1\} \).

- Examples of symbols: \( \Rightarrow, \forall, \exists, \in, \lim, \int, \frac{d}{dx} \).

- Use the correct symbols, logical explanation to present your reasoning (proofs).

- The goal of this course is to provide you the basic training.

- You will need all your old knowledge in algebra, trigonometry, calculus, and also the newly acquired knowledge to do proofs.

Example 1 Real numbers, basic operations and properties (squares are nonnegative, mid-point between two numbers), quadratic equations may have none, one, two real solutions, etc.

Example 2 Calculus, functions, maximization.
Chapter 1 Sets

Definition A set is a collection of objects, its elements (or members).

Notation

We use curly bracket \{\} to embrace the elements in the set, elements are separated by comma ‘,’ in the brackets,

\[ x \in S: x \text{ is an element (member) of } S, \text{ or } x \text{ belongs to the set } S; \]
\[ x \notin S: x \text{ is not an element (member) of } S, x \text{ does not belong to the set } S. \]

\(|S|: \text{ Cardinality (size, or cardinal number) of a set } S, \text{ which may be finite or infinite.} \]

A set is finite if \(|S| = n\) for some nonnegative integer \(n\).

1.1 Describing a set by (a) listing all elements; (b) describing the property.

\( A = \{a, e, i, o, u\}; \) A is the set of vowels; \( a \in A; a, e \in A, m \notin A; |A| = 5. \)

\( B = \{\text{cat, dog, pig}\} \) is the set consisting of the elements: cat, dog, pig; \( |B| = 3. \)

\( C = \{1, 2, \{1, 3\}, \text{cat}\}; \{1, 3\} \in C; \text{cat} \in C; 1 \in C; 3 \notin C. \)

\( D = \{-2, 2\} = \{x: x \text{ is a real number such that } x^2 = 4\} = \{x: x \text{ is a real number such that } |x| = 2\}. \)

\( E = \{1, 3, 5, 7\} = \{x: (x - 1)(x - 3)(x - 5)(x - 7) = 0\}; E \text{ is the set of odd integers between 0 and 8.} \)

Remark Two sets \( A \) and \( B \) are equal, denoted by \( A = B, \) if they contain the same elements.

That is, every element in \( A \) is an element in \( B, \) and vice versa.

Example \( \{1, 2, 3\} = \{1, 3, 2\} = \{1, 2, 2, 3\}. \)
Special sets

Empty set $\emptyset$. It is also called the null set or the void set. Note that $|\emptyset| = 0$.

Example Suppose $F = \{\emptyset, \{\emptyset\}\}$.

(a) $\emptyset \in F$?  
(b) $\{\emptyset\} \in F$?  
(c) $|F| = ?$?

We always assume that there is a universal set $U$ containing all the objects under consideration.

It will lead to a (Russell) paradox if we assume that there is a set containing EVERYTHING.

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is the set of natural numbers.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

$\mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{N}\} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$ is the set of rational numbers.

$\mathbb{R}$ is the set of real numbers.

$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$ is the set of complex numbers.

Remark All of the above sets are infinite, say, $|\mathbb{N}|$ is infinite, or we write $|N| = \infty$.

Note The notation $|S| = \infty$ for a set $S$ simply means that $S$ is infinite (not finite).
1.2 Subsets

Definition A set $X$ is a subset of $Y$ if every element in $X$ is an element in $Y$, denoted by $X \subseteq Y$.

If in addition that $X \neq Y$, then $X$ is a proper subset of $Y$.

Example $X = \{1, 2, 3\}, Y = \{1, 2, 3, 4, 5\}, Z = \{4\}$.

Example $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Remark Two sets $A$ and $B$ are equal if $A \subseteq B$ and $B \subseteq A$.

Definition The set of all subsets of $X$ is the power set of $X$, denoted by $\mathcal{P}(X)$.

Example (a) $X = \emptyset$. (b) $Y = \{1\}$. (c) $Z = \{0, \emptyset, \{\emptyset\}\}$.

Distinction between subsets and memberships

Example Suppose $F = \{\emptyset, \{\emptyset\}\}$. 
1.3 Set operations and Venn diagrams

Venn diagrams can help depict the relationships and operations on sets.

**Definition** Let $X$ and $Y$ be sets.

- Their union, denoted by $X \cup Y$, is the set $\{ x \in U : x \in X \text{ or } x \in Y \}$.
- Their intersection, denoted by $X \cap Y$, is the set $\{ x \in U : x \in X \text{ and } x \in Y \}$.
- The complement of $X$ in the universal set $U$, denoted by $\overline{X}$, is the set $\{ x \in U : x \notin X \}$.
- The relative complement of $X$ in $Y$, denoted by $Y - X$, is the set $\{ x \in U : x \in Y, x \notin X \}$.

**Example** $X = \{1, 2, 3\}, Y = \{1, 2, 3, 4, 5\}, Z = \{4\}.$
Recall the notation of intervals of real numbers.

\[ [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \]

\[ [a, b) = \]

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**Example** \( A = [1, 5], B = [2, 6), C = \{1, 2, 3, 4, 5, 6\} \).
1.4 Indexed collections of sets

We may consider a family of sets $A_j$ with $j$ lying in an index set $J$. Then we can consider their union, intersections, etc.

Example Let $A_r = [0, r]$ with $r > 0$; $B_r = \{0, r\}$.

One may determine $\cap_{r \in R} A_r$, $\cup_{r \in R} A_r$, say, with $R = \{1, 2, 3\}$. 
1.5 Partition of sets

A partition of a set $X$ is a collection of pairwise disjoint nonempty subsets whose union is $X$.

Examples $X = \{1, 2\}$ and $Y = \{\phi, \{\phi\}\}$.

1.6 Cartesian products of sets

Definition Let $A, B$ be sets. Their Cartesian product is the set

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$ 

Examples $\mathbb{N} \times \mathbb{R}$ and $[1, 2] \times [3, 4]$. 