Chapter 10 Functions

Definition let $A, B$ be non-empty sets. A function (map, mapping) $f$ from $A$ to $B$, written as $f : A \rightarrow B$, is a relation from $A$ to $B$ such that every element in $A$ is related to a unique element in $B$.

Terminology The set $A$ is the domain of $f$, $B$ is the co-domain of $f$.

Notation We write $f(a) = b$ if $(a, b) \in f$, and we say that $b$ is the image of $a$ under $f$, also, $f$ maps $a$ to $b$.

Terminology Two maps $f : A \rightarrow B$ and $g : A \rightarrow B$ are equal if $f(a) = g(a)$ for all $a \in A$.

Notation The set of all functions from $A$ to $B$ is the set $B^A = \{f : f$ is a function from $A$ to $B\}$.

Examples $\{a, b, c\}^{\{1,2\}}$, $\mathbb{R}^\mathbb{N}$, etc.
Definition Let \( f : A \to B \) be a function.

It is **one-to-one (injective)** provided \( f(a_1) \neq f(a_2) \) whenever \( a_1 \neq a_2 \) in \( A \),

i.e., if \( f(a_1) = f(a_2) \) then \( a_1 = a_2 \).

It is **onto (surjective)** provided the range of \( f \) is \( B \),

i.e., for every \( b \in B \) there is an \( a \in A \) such that \( f(a) = b \).

It is **bijective (one-one and onto)** if it is both injective and surjective.

Examples

(a) \( f : \mathbb{R} \to \mathbb{R} \), ...

(b) \( f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \) such that \( f(x) = \frac{3x}{x-2} \).

(c) \( f : \mathbb{Z}_4 \to \mathbb{Z}_4 \) such that \( f([x]) = [3x + 1] \).

(d) \( f : \mathbb{Z} \to 2\mathbb{Z} \) such that \( f(x) = 2x \).
**Theorem** Suppose $A$ and $B$ are finite non-empty sets with same number of elements, and $f : A \rightarrow B$. Then $f$ is one-one if and only if $f$ is onto.

**Remark** For $f, g : \mathbb{R} \rightarrow \mathbb{R}$, we can define $f \pm g$, $fg$, and $f/g$ if $g(x)$ is never 0.
**Definition** If $f : A \to B$ and $g : B \to C$, then the **composite** function $h = g \circ f : A \to C$ is defined by $h(a) = g(f(a))$ for every $a \in A$.

**Theorem** Let $f : A \to B$ and $g : B \to C$.

(a) If $f$ and $g$ are one-one, then so is $g \circ f$.

(b) If $f$ and $g$ are onto, then so is $g \circ f$.

(c) If $f$ and $g$ are bijective, then so is $g \circ f$. 
**Definition** Given a relation $R$ from $A$ to $B$, we can define the **inverse relation** $R^{-1}$ from $B$ to $A$.

**Theorem** Let $f : A \rightarrow B$ be a function. then the inverse relation $f^{-1}$ from $B$ to $A$ is a function if and only if $f$ is bijective. In such a case, $f^{-1}$ is also bijective.
Permutations of a set $A$ are bijections from $A$ to $A$. If $A$ has $n$ elements, we assume $A = \{1, \ldots, n\}$, and use a special representation.

Suppose $A = \{1, \ldots, n\}$. We use the notation $S_n$ to denote the set of bijections from $A$ to $A$.

For example, $S_3$ has 6 elements:

\[
\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.
\]

One can find inverses, and do compositions.
Important definitions and examples in function theory.

Let $A, B$ be sets, and $f \subseteq A \times B$ be a relation.

- $f : A \to B$ is a function if every $a \in A$ is related to one and only one element $b = f(a)$ in $B$.

  Examples. $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x - 1$ is not a function; $f : \mathbb{Z}_4 \to \mathbb{Z}_6$ defined by $f([x]_4) = [5x]_6$ is not a function.

- The inverse relation $f^{-1}$ is a function if and only if $f$ is bijective, and $f^{-1} : B \to A$ is the inverse function.

  Note: The notation $f^{-1}$ can represent the inverse relation, the inverse function if $f$ is bijective. When $f$ is a function, $f^{-1}$ also represents the inverse image of $B_1 \subseteq B$ so that $f^{-1}(B_1) = \{x \in A : f(a) \in B_1\}$.

- A function $f : A \to B$ is one-one if $a_1 \neq a_2$ ensures $f(a_1) \neq f(a_2)$. Equivalently, $f(a_1) = f(a_2)$ ensures $a_1 = a_2$.

- A function $f : A \to B$ is onto if $f(A) = B$. Equivalently, for every $b \in B$ we can find $a \in A$ such that $f(a) = b$. 