Non-negative Matrices and Markov Chains

Part I Fundamental Concepts and Results in the Theory of Non-negative Matrices

1. Definition of non-negative matrix and primitive matrix: T is primitive if there exists a positive integer k such that $T^k >$ 0.

<u>*The Perron-Frobenius Theorem for Primitive Matrices*</u>

Suppose T is an n by n non-negative primitive matrix. Then there exists an eigenvalue r such that r is a real positive simple root of the characteristic equation of T, $r > |\lambda|$ for any eigenvalue $\lambda \neq r$ and the eigenvectors associated with r are unique to constant multiples.

(It highlights the nature of finite Markov Chain which is the convergence of an irreducible finite Markov Chain to its stationary distribution.)

2. Behaviour of the powers of T: single self-communicating class is called irreducible.

A non-negative matrix with at least one positive entry in each row possesses at least one essential class of indices.

(Proof: Suppose all indices are inessential. The assumption of non-zero rows then implies that for any index i, i = 1, . . . , n, there is at least one j such that i and j cannot communicate with each other. Now suppose i, is any index. Then we can find a sequence of indices i_2, i_3, . . . such that $i_k \to i_{k+1}$, but i_{k+1} not $\to i_k$. However, sincethe sequence i_2, i_3, . . . i_{n+1}, is a set of n + 1 indices, each chosen from thesame n possibilities, 1,2, . . . , n, at least one index repeats in the sequence. This is a contradiction to the deduction that no index can lead to an index with a lower subscript.)

Part II Markov Chains and Finite Stochastic Matrices

1. Concepts

   If the system is in the ith state at time k - 1, the next jump will take it to the jth state with probability pij(k), which means *future probabilistic evolution of the process is determined once the immediate past is known*.

2. Classification

   Homogeneous: stationary transition probabilities.

Example: *Bernoulli scheme: We denote success with the number 1 and failure with the number 0; then the transition matrix at any time k is* $\begin{pmatrix} p & q \\ p & q \end{pmatrix}$.

Non-homogeneous:

Example: *Polya Urn scheme: Imagine we have a white and b black balls in an urn. Let a + b = N. We draw a ball at random and before drawing the next ball we replace the one drawn, adding also s balls of the same colour. Let us say that after r drawings the system is in state i, which is the number of white balls obtained in the r drawings. Suppose we are in state i after drawing number r. Thus r - i black balls have been drawn, and the number of white balls in the urn is a + is, and the number of black is b + (r - i)s. Then at the next drawing we have movement to state i + 1 with probability* $\frac{a+is}{N+rs}$.

Part III. Limiting Probabilities

1. Calculation of P^n

   Example: diagonalizable

   Diagonalize P = $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$, direct calculation from linear algebra we find out

$$\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lim_{n\to\infty} \mathrm{P}^n = \lim_{n\to\infty} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} (-\frac{1}{4})^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

$$(I - zP)^{-1} = (1 + z + z^2 + z^3 + \cdots) \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix} + [1 - \frac{z}{4} + (\frac{z}{4})^2 - (\frac{z}{4})^3 + \cdots] \begin{pmatrix} \frac{2}{5} & -\frac{2}{5} \\ -\frac{3}{5} & \frac{3}{5} \end{pmatrix}$$

$$P^n = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix} + (-\frac{1}{4})^n \begin{pmatrix} \frac{2}{5} & -\frac{2}{5} \\ -\frac{3}{5} & \frac{3}{5} \end{pmatrix}, \quad \lim_{n\to\infty} \mathrm{P}^n = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

2. Stationary distribution

$\pi \mathrm{P} = \pi$

Example: $\mathrm{P} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $\pi = (\frac{a_{21}}{a_{12} + a_{21}} \quad \frac{a_{12}}{a_{12} + a_{21}})$

References

E. Seneta. *Non-negative Matrices and Markov Chains (Springer Series in Statistics)*. Publication Date: January 26, 2006, Edition: 2nd.

Kiyoshi Igusa. *Notes on Stochastic Processes*. [electronic resource]. lecture notes from an undergraduate course given at Brandeis University in Fall 2006 using the second edition of Gregory Lawler's book *Introduction to Stochastic Processes*. December 17, 2006. Available from: http://people.brandeis.edu/~igusa/Math56F06/Math56a_lectures.pdf

James E. Gentle. *Matrix Algebra: Theory, Computations, and Applications in Statistics (Springer Texts in Statistics)*. Publication Date: July 27, 2007. Edition: 1.

D. Pollrard. *Convergence of Stochastic Processes (Springer Series in Statistics)*. October 8, 1984. 1 edition.