Math 300 – Mathematical Sciences Writing

A Mathematical Introduction to Quantum Sensing

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1. Introduction

Recent technological advances allow for individual quanta to be controlled in a variety of physical quantum systems. This has spurred the development of dedicated systems to implement quantum computing and quantum communications. The quantum properties of these systems allow for performance that is impossible to achieve classically, like exponential faster algorithms for certain problems and perfectly information-theoretically secure communications. Another important technology that has developed in parallel with quantum computing and communications is that of quantum sensing.

While one of the major difficulties in physically implementing a quantum computer is isolating the system from its environment due to the quantum system's inherently high sensitivity to external stimuli, this high sensitivity is hugely beneficial for sensing applications. Quantum sensing utilizes a quantum system embedded in an environment that senses some stimulus from the environment by measureing how the system evolves in response to the stimulus. Quantum sensing as a field is still in its early stages, but realized benefits over classical sensing include higher sensitivies [Swithenby, 1987], being able to use small sensing volumes to probe features on the sub-micron scale [Kucsko et al., 2013], smaller SWaP (size, weight, and power) [Cox et al., 2018], and better tracibility to fundamental constants (e.g. $\hbar, c$) [Anderson et al., 2019]. This in turn has lead to applications such as imaging brain activity by measuring the induced magnetic fields [Swithenby, 1987], taking the temperature of organisms at the sub-cellular level [Kucsko et al., 2013], creating efficient RF receivers that are much smaller and more efficient than traditional antannae [Cox et al., 2018], and better metrology through SI-traceable RF receivers in domains where only relative calibrations were previously possible [Anderson et al., 2019]. As the field matures, many more applications are likely to emerge.

2. Mathematical Formulation of Quantum Mechanics

Before discussing the mathematics of quantum sensing, it is worthwhile to first discuss how mathematics, particularly matrix mechanics, can be used to describe quantum mechanical systems.

2.1 The Wavefunction

One of the basic ideas of quantum mechanics is that quantum objects are not localized, but can be represented by some probability density function $f(\mathbf{r})$ of position ($\mathbf{r}$ represents a position vector). More commonly, a particle is considered to be in a 'state' given by a wavefunction $|\psi(\mathbf{r})\rangle$ in a complex Hilbert space $\mathcal{H}$, where the probability density function f is now given by the modulus of wavefunction squared, i.e.

$$f(\mathbf{r}) = |\ |\psi(\mathbf{r})\rangle\ |^2.$$

This method of writing wavefunctions is known as the Dirac notation. The state $|\psi\rangle$ is known as a 'ket' vector when written in this way, and is a column vector when $\mathcal{H}$ is finite-dimensional. When describing quantum states this way, it is customary to extend $\mathcal{H}$ to define a dual vector space that also contains the vector space of the form

$$\mathcal{H}^\dagger = \{\langle\psi| : |\psi\rangle^\dagger\}$$

where † denotes taking the complex conjugate and transpose (in either order since these operations commute) and the $\mathbf{r}$-dependence of $|\psi\rangle$ is dropped for convenience. These vectors are called 'bra' vectors, and are row vectors when $\mathcal{H}$ is finite-dimensional. These names come about as Dirac notation is also referred to as 'bra-ket' (bracket) notation.

The construction of this dual vector space allows for an efficient process and notation for inner products in $\mathcal{H}$. We define an inner product on the ket vector space $\mathcal{H}$ as

$$|\psi_1\rangle \cdot |\psi_2\rangle = \langle\psi_1|\psi_2\rangle.$$

Since the probability density function f must integrate over all space to 1, any valid wavefunction must be normalized as

$$1 = \langle\psi(\mathbf{r})|\psi(\mathbf{r})\rangle$$

In general, the Hilbert space $\mathcal{H}$ can be a vector space of any dimension, including infinite, but for the purposes of most applications in quantum information science, including discussing quantum sensing later, we can choose $\mathcal{H}$ to be a two dimensional vector space, so $\mathcal{H} = \mathbb{C}^2$. In this case, we can discuss 'operators', which are mathematical objects that act on quantum states to change them, as complex 2×2 matrices. We customarily denote them with a hat like $\hat{O}$.

A particularly important class of operators are Hermitian operators, which satisfy $\hat{O} = \hat{O}^\dagger$. Such operators have eigenvalues which correspond to physical observable quantities that can be measured in a laboratory. It can also be shown (as in Theorem 1.3 on page 17 of [Nakahara and Ohmi, 2008]) that eigenvectors corresponding to different eigenvalues of a Hermitian operator form an orthonormal basis, and the Gram-Schmidt process can be used to create an orthonormal basis even with degenerate eigenvalues. We usually denote such a basis as $\{|0\rangle, |1\rangle\}$.

2.2 The Hamiltonian

One particularly important operator in quantum mechanics is the Hamiltonian operator $\hat{H}$, which has eigenvalues that correspond to the possible measurable energies of a quantum particle or system. The Hamiltonian is used to define the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

which governs many quantum mechanical systems[1] by describing what sorts of solutions are possible and how they evolve with respect to time.

If we choose our orthonormal basis $|0\rangle, |1\rangle$ to be the eigenvectors of $\hat{H}$ corresponding to the eigenvalues (energies) of $\hat{H}$: E_0, E_1, then we can write any $|\psi\rangle \in \mathcal{H}$ as

$$|\psi\rangle = a|0\rangle + b|1\rangle \qquad \text{for } a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1$$

since

$$1 = \langle\psi|\psi\rangle = (a|0\rangle + b|1\rangle)^\dagger(a|0\rangle + b|1\rangle) = (a^*\langle 0| + b^*\langle 1|)(a|0\rangle + b|1\rangle)$$
$$= aa^*\langle 0|0\rangle + bb^*\langle 1|1\rangle + ab^*\langle 1|0\rangle + a^*b\langle 0|1\rangle = aa^* + bb^* = |a|^2 + |b|^2$$

because $|0\rangle, |1\rangle$ form an orthonormal basis (using $*$ to denote complex conjugation).

Another of the central tenants of quantum mechanics is that measurements are inherently invasive. If we take $|\psi\rangle = a|0\rangle + b|1\rangle$ as above, and then measure the energy of the state described by $|\psi\rangle$, we get a result of E_0 with probability $|a|^2$ and E_1 with probability $|b|^2$. As the measurement is taken, the state $|\psi\rangle$ collapses to $|0\rangle$ if E_0 is measured and $|1\rangle$ if E_1 is measured [Nakahara and Ohmi, 2008]. That is, after measuring either E_0 or E_1 the first time, any susbsequent measurement will always return the exact same result.

A useful concept that we'll use later is that of spectral decomposition. We define the spectral decomposition of a matrix/operator as

$$\hat{O} = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|$$

where $\{\lambda_j\}$ is an ordered set of eigenvalues of $\hat{O}$ corresponding to the eigenvectors $\{|\lambda_j\rangle\}$. For example, if we let $|0\rangle, |1\rangle$ be the standard basis $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, then the spectral decomposition of $\hat{H}$ is

$$\hat{H} = E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1|$$

which also gives us the matrix form of $\hat{H}$

$$\hat{H} = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}.$$

[1] The Schrödinger Equation is only valid for nonrelativistic quantum systems. Accounting for special relativity gives rise to the Dirac Equation, and trying to account for general relativity gives rise to quantum field theory. Thankfully, we can avoid the complexities of these cases as most quantum sensing systems are well-described by the Schrödinger Equation.

When discussing the Hamiltonian, it is also useful to discuss the time-evolution operator defined as

$$\hat{U}(t_1, t_2) = e^{\frac{i}{\hbar}\int_{t_1}^{t_2} \hat{H} d\tau}$$

which time-evolves a solution of the Schrödinger Equation $|\psi(\mathbf{r}, 0)\rangle$ from a time t_1 to a time t_2. That is,

$$\hat{U}(0, t)|\psi(\mathbf{r}, 0)\rangle = |\psi(\mathbf{r}, t)\rangle$$

In the case where the Hamiltonian is constant with respect to time, we can fix any t_1 as $t = 0$, and then

$$\hat{U}(0, t) = e^{\frac{i}{\hbar}\int_0^t \hat{H} d\tau} = e^{\frac{it}{\hbar}\hat{H}}$$

which also allows for $\hat{U}(t)$ to be written down in its spectral decomposition using the spectral decomposition of $\hat{H}$ above. In particular,

$$\hat{H} = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j| = E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1|$$

$$\Rightarrow \hat{U}(0, t) = \sum_j e^{\frac{it}{\hbar}\lambda_j}|\lambda_j\rangle\langle\lambda_j| = e^{\frac{it}{\hbar}E_0}|0\rangle\langle 0| + e^{\frac{it}{\hbar}E_1}|1\rangle\langle 1| = \begin{pmatrix} e^{\frac{it}{\hbar}E_0} & 0 \\ 0 & e^{\frac{it}{\hbar}E_1} \end{pmatrix} = e^{\frac{it}{\hbar}E_0}\begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{it}{\hbar}(E_1 - E_0)} \end{pmatrix}$$

where we make use of the fact that the time-evolution operator is analytic in this case. Further explanation of this technique can be found on page 352 of [Griffiths, 1995]. The important result to focus on here is the final matrix representation of the time-evolution operator. We can clearly see that two states with different energies will pick up a relative phase factor that depends on time, and this phase difference is the crux of the general quantum sensing algorithm that will developed later.

2.3 Using Complex Numbers

As discusssed previously, we can choose our Hilbert space $\mathcal{H}$ as $\mathbb{C}^2$, and for a Hermitian operator $\hat{O}$ with (real) eigenavalues λ_0, λ_1 with associated eigenvectors $|0\rangle, |1\rangle \in \mathcal{H}$, then $|0\rangle, |1\rangle$ is a basis for $\mathcal{H}$ and we can write any $|\psi\rangle \in \mathcal{H}$ as

$$|\psi\rangle = a|0\rangle + b|1\rangle \qquad \text{for } a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1$$

where taking a physical measurement of the state corresponding to $\hat{O}$ results in λ_0 with proability $|a|^2$ and λ_1 with probability $|b|^2$. For this reason, introducing an imaginary phase factor $e^{i\theta}$ results in

$$e^{i\theta}|\psi\rangle = ae^{i\theta}|0\rangle + be^{i\theta}|1\rangle$$

and the probability of measuring $|0\rangle$ is $|ae^{i\theta}|^2 = |a|^2|e^{i\theta}|^2 = |a|^2$ and similarly $|b|^2$ for $|1\rangle$. Thus, we can see that any complex phase factor of the form $e^{i\theta}$ does not effect the readout of experiment. In fact, there is no experimental way to determine what such a θ might be, so it is often customary to consider

$$|\psi\rangle = e^{i\theta}|\psi\rangle \qquad \text{for any } \theta \in [0, 2\pi)$$

as a leading phase factor means nothing for any real-world application.

At this point, it is also worth discussing the topic of the Pauli Matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which, along with the identity matrix I_2, forms a basis for $M_2(\mathbb{C})$, the ring of all 2×2 matrices with elements in $\mathbb{C}$. These will be used later to rewrite matrices in an intuitive way and to make use of external results to simplify some calculations.

2.4 The Tensor Product

So far we have discussed the mathematics of a single quantum object or bit, called a qubit, but what happens when we would like to consider multiple qubits? To do this, we must introduce the tensor product. In general, a tensor product allows one to describe a combined system. When we are dealing with finite-dimensional vector spaces, we can define a tensor product using the Kronecker product. For an $m \times n$ matrix A and an $a \times b$ matrix B, the Kronecker product $A \otimes B$ is an $ma \otimes nb$ block matrix, where there are $m \times n$ blocks of size $a \times b$ and the i, j block is $(A)_{ij}B$. For example, for

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} (1)B & (2)B \\ (3)B & (4)B \end{pmatrix} = \begin{pmatrix} 5 & 6 & 10 & 12 \\ 7 & 8 & 14 & 16 \\ 15 & 18 & 20 & 24 \\ 21 & 24 & 28 & 32 \end{pmatrix}$$

although A and B need not be the same shape, or have any similar dimensions, in general.

Now to apply this to our quantum mechanical systems. Let us have a first particle in a state $|\psi_1\rangle \in \mathcal{H}_1$ and a second particle in state $|\psi_2\rangle \in \mathcal{H}_2$. Then we can consider the overall state $|\psi\rangle$ as

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

We call such a state a 'separable' state, as the state $|\psi\rangle$ can be written as the tensor product of two individual states each located in their own Hilbert space. However, the combined space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is closed under finite sums, and so also contains 'entangled' or 'inseparable' states such as $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$, where we assume $|0\rangle, |1\rangle$ is a basis for both $\mathcal{H}_1$ and $\mathcal{H}_2$. These states are inseparable as there are no $|\psi_1\rangle \in \mathcal{H}_1, |\psi_2\rangle \in \mathcal{H}_2$ such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$.

Proof. By way of contradiciton, suppose there was such a $|\psi_1\rangle = a|0\rangle + b|1\rangle \in \mathcal{H}_1, |\psi_2\rangle = c|0\rangle + d|1\rangle \in \mathcal{H}_2$ such that $|\psi_1\rangle \otimes |\psi_2\rangle = |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$. Then $|\psi_1\rangle \otimes |\psi_2\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) =$ $ac|0\rangle \otimes |0\rangle + ad|0\rangle \otimes |1\rangle + bc|1\rangle \otimes |0\rangle + bd|1\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$. Then $ac = bd = \frac{1}{\sqrt{2}}$ and $ad = bc = 0$. But then one each of a, d and b, c must be zero, which means at least one of ac or bd is zero, a contradiction. Hence, $|\psi\rangle$ is inseparable. □

While separable states can be described in purely classical manners, inseparable/entangled states are purely quantum mechanical phenomenon, and utilizing entangled states is what makes quantum computing, communications, and sensing more powerful than their classical counterparts.

3. General Quantum Sensing Protocol

We wish to measure some external stimulus which we'll denoted by a complex-valued function of time $V(t)$. To do this, the general procedure is to initialize the quantum system to a known quantum state, expose it to the stimulus which changes the state, wait as the system evolves for some time t, make some measurement of the new state to get some information about the stimulus (which collapses the wavefunction to an eigenvector of the chosen observable), and then repeat this process to try to get more information about the stimulus. We'll loosely follow the discussion in [Degen et al., 2017], although with deeper explanation and emphasis of the basics to make things more approachable.

Assumptions

We assume that there is a quantum system with two distinct states, and for the purposes of this discussion, we assume that the two states $|0\rangle, |1\rangle$ are eigenvectors of the sensor's natural Hamiltonian and hence correspond to energies E_0, E_1, respectively, with some difference in energy $\Delta E = E_1 - E_0$ between them. Also, we'll assume we're operating in the limit where the energies E_0, E_1, and ΔE are much larger than the changes in the Hamiltonian introduced by $V(t)$, which is a valid assumption for most quantum sensing applications per [Degen et al., 2017].

The Sensing Hamiltonian

A quantum sensor's Hamiltonian $\hat{H}$ can be broken into components as

$$\hat{H}(t) = \hat{H}_0(t) + \hat{H}_V(t) + \hat{H}_{control}(t)$$

where $\hat{H}_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}$ is the Hamiltonian of the system in the absence of external stimulus, $\hat{H}_V(t)$ is the change in the Hamiltonian due to the external stimulus, and $\hat{H}_{control}(t)$ is used for tuning the separation of the energy levels and implementing unitary operations with quantum gates as required to carry out a sensing protocol [Degen et al., 2017, Nakahara and Ohmi, 2008]. If we assume that $\hat{H}_{control}(t)$ is 0 for now, the Hamiltonian $\hat{H}(t) = \hat{H}_0(t) + \hat{H}_V(t)$ can be represented by a 2×2 matrix

$$\hat{H}(t) = \begin{pmatrix} E_0 + a(t) & b(t) \\ c(t) & E_1 + d(t) \end{pmatrix} = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} + \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} = \hat{H}_0(t) + \hat{H}_V(t)$$

Recall that the Hamiltonian is a Hermitian matrix, hence $\hat{H}(t)^\dagger = \hat{H}(t)$ and so

$$\begin{pmatrix} E_0 + a(t) & b(t) \\ c(t) & E_1 + d(t) \end{pmatrix} = \begin{pmatrix} E_0 + a(t)^* & c(t)^* \\ b(t)^* & E_1 + d(t)^* \end{pmatrix}$$

which results in $a(t) = a(t)^*, d(t) = d(t)^*$, so $a(t), d(t)$ are real-valued functions, and $b(t)^* = c(t)$, so $b(t), c(t)$ are complex conjugates. We can now rewrite as

$$\hat{H}(t) = \hat{H}_0(t) + \hat{H}_V(t) = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} + \begin{pmatrix} a(t) & c(t)^* \\ c(t) & d(t) \end{pmatrix}$$

One final manipulation can be made to remove $\frac{a(t)+d(t)}{2} I$ from the second matrix. While this is slightly abusive[2], this allows us to rewrite

$$\hat{H}_V(t) = \begin{pmatrix} a(t) & c(t)^* \\ c(t) & d(t) \end{pmatrix} \rightarrow \begin{pmatrix} \frac{a(t)-d(t)}{2} & c(t)^* \\ c(t) & -(\frac{a(t)-d(t)}{2}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\gamma V_\parallel(t) & \frac{1}{2}\gamma V_\perp(t)^* \\ \frac{1}{2}\gamma V_\perp(t) & \frac{1}{2}\gamma V_\parallel(t) \end{pmatrix}$$

where we introduce γ as the transduction parameter/function that relates changes in the external stimulus to changes in the Hamiltonian, and $V_\parallel(t)$ and $V_\perp(t)$ are the components of $V(t)$ that represent changes in the energy levels and the couplings between them, as determined by their positions on and off of the main diagonal, respectively [Degen et al., 2017]. In practical, physical implementations, V is commonly a magnetic or electric field, and γ is commonly a linear or differential coupling of the sensor's Hamiltonian to that field which also makes the units of $\gamma V(t)$ units of energy. It is worth mentioning that the negative sign is chosen for the upper-right $\frac{1}{2}\gamma V_\parallel(t)$ element as when $V_\perp = 0$,

$$\hat{H}(t) = \hat{H}_0(t) + \hat{H}_V(t) = \begin{pmatrix} E_0 - \frac{1}{2}\gamma V_\parallel(t) & 0 \\ 0 & E_1 + \frac{1}{2}\gamma V_\parallel(t) \end{pmatrix}$$

which means the stimulus $V_\parallel$ causes the difference in energy between the E_1 and E_0 to increase by $\gamma V_\parallel$.

Another benefit of using this rewriting is that can decompose $\hat{H}_V$ as

$$\hat{H}_V(t) = \begin{pmatrix} -\frac{1}{2}\gamma V_\parallel(t) & \frac{1}{2}\gamma V_\perp(t)^* \\ \frac{1}{2}\gamma V_\perp(t) & \frac{1}{2}\gamma V_\parallel(t) \end{pmatrix} = \frac{1}{2}\gamma \begin{pmatrix} -V_\parallel(t) & \Re[V_\perp(t)] - i\Im[V_\perp(t)] \\ \Re[V_\perp(t)] + i\Im[V_\perp(t)] & V_\parallel(t) \end{pmatrix}$$

$$= \frac{1}{2}\gamma[\Re[V_\perp(t)]\sigma_x + \Im[V_\perp(t)]\sigma_y - V_\parallel(t)\sigma_z]$$

where $\Re$ and $\Im$ denote taking the coefficient of the real and imaginary components of a complex number or valued-function, and we make use of the Pauli matrices discussed earlier. In this way, the external stimulus $V(t)$ induces components $\frac{1}{2}\gamma\Re[V_\perp(t)]$, $\frac{1}{2}\gamma\Im[V_\perp(t)]$, and $-\frac{1}{2}\gamma V_\parallel(t)$ that are analogous to changes in the spin in the x, y, and z directions, respectively.

Now let's define

$$\hat{H}_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} = \begin{pmatrix} E_{avg} - \Delta E/2 & 0 \\ 0 & E_{avg} + \Delta E/2 \end{pmatrix} = E_{avg} I - (\Delta E/2)\sigma_z$$

[2]If it makes you feel better, consider that we can absorb it into the control Hamiltonian $\hat{H}_{control}$ to achieve the same effect. Also, we saw earlier when discussing the time-evolution operator that the relative difference in energies is more important than the magnitude of the energies.

for $E_{avg} = (E_0 + E_1)/2$. Then we can express $\hat{H}$ as

$$\hat{H} = \hat{H}_0 + \hat{H}_V = E_{avg}I + (\frac{1}{2}\gamma\Re[V_\perp(t)])\sigma_x + (\frac{1}{2}\gamma\Im[V_\perp(t)])\sigma_y - (\frac{1}{2}\gamma V_\parallel(t) + \frac{1}{2}\Delta E)\sigma_z$$

which, through simple calculation, can be shown to have eigenvalues of

$$\lambda_\pm = E_{avg} \pm \frac{\Delta E}{2}\sqrt{1 + 2\frac{\gamma V_\parallel}{\Delta E} + \frac{\gamma^2}{\Delta E^2}(V_\parallel^2 + \Re[V_\perp]^2 + \Im[V_\perp]^2)}$$

In the limit where $\Delta E >> |V_\parallel|, |V_\perp|$, which we assumed earlier, we can Taylor approximate this as

$$\lambda_\pm \approx E_{avg} \pm \frac{\Delta E}{2}(1 + \frac{\gamma V_\parallel}{\Delta E} + \frac{\gamma^2}{2\Delta E^2}(V_\parallel^2 + \Re[V_\perp]^2 + \Im[V_\perp]^2))$$

$$= E_{avg} \pm (\frac{\Delta E}{2} + \frac{1}{2}\gamma V_\parallel + \mathcal{O}(\frac{1}{\Delta E})) \approx E_{avg} \pm (\frac{\Delta E}{2} + \frac{1}{2}\gamma V_\parallel)$$

$$= E_1 + \frac{1}{2}\gamma V_\parallel(t), E_0 - \frac{1}{2}\gamma V_\parallel(t)$$

This then admits a spectral decomposition of $\hat{H}$ as

$$\hat{H}_0 = (E_0 - \frac{1}{2}\gamma V_\parallel(t))|\lambda_0\rangle\langle\lambda_0| + (E_1 + \frac{1}{2}\gamma V_\parallel(t))|\lambda_1\rangle\langle\lambda_1|$$

for $|\lambda_0\rangle, |\lambda_1\rangle$ the eigenvalues of $\hat{H}$ corresponding to $E_0 - \frac{1}{2}\gamma V_\parallel(t), E_1 + \frac{1}{2}\gamma V_\parallel(t)$. With this discussion and analysis of the sensing Hamiltonian complete, we can dive into the general quantum sensing algorithm.

3.1 Initialization

To initialize a given system to a known quantum state, we first start with a known ground state $|0\rangle$. Then, depending on the type of information that we want to learn about the stimulus, the measurement scheme to be used, and the physical implementation of the quantum system, we choose some unitary operator U_{Init} such that it transforms our state $|0\rangle$ to a desired initial superposition state $|\psi_{Init}\rangle = a|0\rangle + b|1\rangle$ for some $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$.

3.2 Effects of Stimulus

After the sensing state is initialized, it is exposed to the environment and evolves according to the time-evolution operator of the sensing Hamiltonian $\hat{U}_H$ as

$$|\psi(t)\rangle = \hat{U}_H(0,t)|\psi_{Init}(0)\rangle$$

In general, $\hat{U}_H(0,t) = e^{\frac{i}{\hbar}\int_0^t \hat{H}d\tau}$ could be a complicated, non-analytical function for a time-dependent $V(t)$ (making $\hat{H}$ time-dependent as well).

However, in the case where $V(t)$ is constant or changes much more slowly than our sensing integration time, we can assume

$$|\psi(t)\rangle = \hat{U}_H(0,t)|\psi_{Init}(0)\rangle = e^{\frac{it}{\hbar}\hat{H}}|\psi_{Init}(0)\rangle$$

which means the sensing state evolves as

$$|\psi(t)\rangle = (e^{\frac{it}{\hbar}(E_0-\frac{1}{2}\gamma V_\parallel)}|\lambda_0\rangle\langle\lambda_0| + e^{\frac{it}{\hbar}(E_1+\frac{1}{2}\gamma V_\parallel)}|\lambda_1\rangle\langle\lambda_1|)|\psi_{Init}(0)\rangle$$

using spectral decomposition and the final representation of the sensing Hamiltonian found previously.

3.3 Readout

After the sensing state has time-evolved in the presence of $V(t)$, it can be transformed again before a measurement is taken. The first part, the transformation to some desired read-out state, is performed by an operator $\hat{U}_{Readout}$ (which is often $\hat{U}_{Init}^{-1}$[Degen et al., 2017]) where

$$|\psi_{Final}\rangle = \hat{U}_{Readout}|\psi(t)\rangle$$

A measurement of this final state $|\psi_{Final}\rangle = a'|0\rangle + b'|1\rangle$ is made with respect to the basis $\{|0\rangle, |1\rangle\}$,[3] where $|0\rangle$ is measured with proability $|\langle 0|\psi_{Final}\rangle|^2 = |a'|^2$ and $|1\rangle$ is measured with probability $|\langle 1|\psi_{Final}\rangle|^2 = |b'|^2$. After this measurement, the sensing state has been collapsed into one of the basis states, so no more information can be gained. However, by having multiple quantum sensing elements time-evolving together or by repeating the process many times before the external stimulus $V(t)$ can change, a transition probability $p_{|0\rangle\to|1\rangle} = |\langle 1|\psi_{Final}\rangle|^2 = |b'|^2$ can be estimated. The 'sensing' is then accomplished by taking a series of these transition probabilities as a time-series, and then using the results to estimate the sensed stimulus $V(t)$ [Degen et al., 2017].

Example

The simplest mathematical example of quantum sensing is sensing an external stimulus's effect on the splitting of the energy levels of an isolated system. Suppose our stimulus is constant and 'parallel' with our sensor, i.e. $V_\parallel(t) = V_0$ and $V_\perp = 0$, and we choose our initialization and readout preparation operators to be the famous Hadamard gate

$$\hat{U}_{Had} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

since $\hat{U}_{Had}^{-1} = \hat{U}_{Had}$.

Then the initial state

$$|\psi_{Init}\rangle = \hat{U}_{Had}|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

[3] This need not necessarily be the same basis into which the system was initialized, but we'll assume it is so that we only have to keep track of one basis.

which time evolves as

$$|\psi(t)\rangle = (e^{\frac{it}{\hbar}(E_0-\frac{1}{2}\gamma V_\parallel)}|0\rangle\langle 0| + e^{\frac{it}{\hbar}(E_1+\frac{1}{2}\gamma V_\parallel)}|1\rangle\langle 1|)|\psi_{Init}(0)\rangle = \begin{pmatrix} e^{\frac{it}{\hbar}(E_0-\frac{1}{2}\gamma V_\parallel)} & 0 \\ 0 & e^{\frac{it}{\hbar}(E_1+\frac{1}{2}\gamma V_\parallel)} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{it}{\hbar}(E_0-\frac{1}{2}\gamma V_\parallel)} \\ e^{\frac{it}{\hbar}(E_1+\frac{1}{2}\gamma V_\parallel)} \end{pmatrix} = \frac{1}{\sqrt{2}} e^{\frac{it}{\hbar}(E_0-\frac{1}{2}\gamma V_\parallel)} \begin{pmatrix} 1 \\ e^{\frac{it}{\hbar}(E_1-E_0+\gamma V_\parallel)} \end{pmatrix}$$

This is then prepared for readout as

$$|\psi_{Final}\rangle = \hat{U}_{Had}|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} e^{\frac{it}{\hbar}(E_0-\frac{1}{2}\gamma V_\parallel)} \begin{pmatrix} 1 \\ e^{\frac{it}{\hbar}(E_1-E_0+\gamma V_\parallel)} \end{pmatrix}$$

$$= \frac{1}{2} e^{\frac{it}{\hbar}(E_0-\frac{1}{2}\gamma V_\parallel)} \begin{pmatrix} 1 + e^{\frac{it}{\hbar}(E_1-E_0+\gamma V_\parallel)} \\ 1 - e^{\frac{it}{\hbar}(E_1-E_0+\gamma V_\parallel)} \end{pmatrix}$$

Then the transition probability

$$p_{|0\rangle\to|1\rangle} = |\langle 1|\psi_{Final}\rangle|^2 = |1 - e^{\frac{it}{\hbar}(E_1-E_0+\gamma V_\parallel)}|^2 = \frac{1}{2}(1 - \cos(t\frac{(E_1 - E_0) + \gamma V_\parallel}{\hbar}))$$

We know the difference in energy between E_1 and E_0, either since we constructed the system or by taking measurements without the external stimulus V, and we can control the time t for which the system is allowed to evolve under the external stimulus. Then we can fix t and take many measurements to estimate $p_{|0\rangle\to|1\rangle}$, which then makes finding $t\frac{\gamma V_\parallel}{\hbar}$ a simple phase-estimation problem which gives us $\gamma V_\parallel$. The physical implementation of this process is known as Ramsey Interferometry, and it can be done with arbitary initialization and readout preparation unitary operators.

4. Benefits of Entanglement

Up until now, I've said that we take many measurements of $|\psi_{Final}\rangle$ to estimate $p_{|0\rangle\to|1\rangle}$, but I've been glossing over the estimation process. Assuming we can take N measurements, either by having N experimental apparatuses running in parallel or by taking N different measurements of a (relatively) constant V with a single apparatus, the uncertainty in p, denoted as σ_p (this is a positive real number; not to be confused with the Pauli matrices), scales as

$$\sigma_p \propto \frac{1}{\sqrt{N}}$$

and if we consider the Ramsey Intferometry example above, then the uncertainty in $\gamma V_\parallel$ and so in $V_\parallel$, denoted σ_V, scales as

$$\sigma_V \propto \sigma_p \propto \frac{1}{\sqrt{N}}$$

This relationship is known as the standard quantum limit (SQL) [Giovannetti et al., 2011], but can also be explained with the Law of Large Numbers from statistics, where measuring

N similarly distributed, well-behaved random variables gives the sample mean as an estimator for the population mean and the sample variance divided by the size of the sample as an uncertainty in the estimate of the population mean.

However, the nature of quantum systems allows for more information to be extracted by exploiting entanglement between quantum systems. This is the fundamental basis for the benefits of quantum computing over classical computing, and quantum sensing has similar benefits over classical sensing. Suppose we return to the Ramsey Interferometry example above, but rather than initializing N sensing qubits separately, we initialize $\frac{N}{n}$ groups each with n entangled quantum systems. Then we have

$$|\psi_{Init}\rangle = \frac{1}{\sqrt{2^n}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$$

where $|0\rangle^{\otimes n} = |0\rangle \otimes \ldots \otimes |0\rangle$, n times. After initialization, each of the n sensing qubits evolves to pick up a relative phase factor of $e^{\frac{it}{\hbar}(E_1 - E_0 + \gamma V_\parallel)}$, which combined results in

$$|\psi(t)\rangle = \mathcal{N}(|0\rangle^{\otimes n} + (e^{\frac{it}{\hbar}(E_1 - E_0 + \gamma V_\parallel)}|1\rangle)^{\otimes n}) = \mathcal{N}(|0\rangle^{\otimes n} + e^{n\frac{it}{\hbar}(E_1 - E_0 + \gamma V_\parallel)}|1\rangle^{\otimes n})$$

where $\mathcal{N}$ is just a factor to take care of normalization. Then the transition probability

$$p_{|0\rangle \to |1\rangle} = |\langle 1|\psi_{Final}\rangle|^2 = |1 - e^{n\frac{it}{\hbar}(E_1 - E_0 + \gamma V_\parallel)}|^2 = \frac{1}{2}(1 - \cos(t\frac{n(E_1 - E_0) + n\gamma V_\parallel}{\hbar}))$$

From this, we can see that through entangling n sensing qubits, the 'signal' we're trying to sense increases from $V_\parallel \to nV_\parallel$, and with $\frac{N}{n}$ total measurements,

$$\sigma_V \propto \frac{1}{n}\sigma_p \propto \frac{1}{n}(\frac{1}{\sqrt{\frac{N}{n}}}) = \frac{1}{\sqrt{Nn}}$$

which means the error decreased by a factor of $\sqrt{n}$. In the case where $n = N$, the uncertainty now scales as

$$\sigma_V \propto \frac{1}{N}$$

which is known as the Heisenberg limit, and is the quantum-mechanically limited, maximal amount of information one can get from taking n quantum sensing measurements [Giovannetti et al., 2011].

5. Conclusion/Summary

Quantum sensing is an emerging field with a large number of exciting applications. In terms of physical implementation, quantum sensors are able to naturally be more sensitive, have better SWaP, and sense on smaller scales than many classical sensors, while also often being directly tracable to fundamental physical constants. Additionally, as we've covered in this paper, after developing the necessary background, even in the simplest sensing algorithm, Ramsey Interferometry, exploting entanglement and the quantum nature of these devices allows for fundamentally more information to be collected than is allowed by classical measurements and the SQL. Quantum sensing fundamentally improves upon the physical and theoretical limitations of classical sensing, and it will be interesting to see how and what physical realizations develop to the point of practicality in the near future.

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