1 Introduction

Maxwell’s Equations are the canonical laws governing a classical description of electromagnetic phenomena. Among various applications is the field of plasma physics and the development of nuclear fusion reactors for an environmentally-friendly mode of energy production. With the immense computational complexity of these systems, however, it is necessary to develop more efficient algorithms to simulate these physical processes. One area of ongoing research devoted to this task is the development Quantum Lattice Algorithms. These algorithms use quantum bits (qubits) rather than classical bits to discretely represent the evolution of a system along a lattice. The goal herein will be to understand and explain the process behind which these algorithms work, rather than developing them further per se.

2 Background

2.1 Qubits

Whereas a classical bit has two possible values, namely 0 and 1, a qubit is a superposition of two basis states \( |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \) with the requirement \( |\alpha|^2 + |\beta|^2 = 1 \).
Due to quantum effects, a qubit remains in this superposition until decoherence or a measurement is taken that causes it to collapse into one of the two basis states. While still in a superposition, however, it has probabilities $|\alpha|^2$ and $|\beta|^2$ of being found in each state, respectively, when a measurement is performed. A qubit can be further expressed using spherical coordinates as $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$ with $\theta \in [0, \pi]$ as the angle from the positive z-orientation and $\phi \in [0, 2\pi]$ as the azimuthal angle. This form allows a graphical representation known as the Bloch sphere.

Figure 1: Caption

For more complex systems (i.e. more qubits), the ability of a qubit to exist in an infinite number of possible states prior to measurement allows significant advantages over the classical bit.

2.2 Maxwell’s Equations

Maxwell’s Equations consist of four vector equations governing all electromagnetic phenomena. Named after James Clerk Maxwell, who consolidated all previously developed equations and studied phenomena into a theory of electromagnetism, the four equations in a dielectric medium are:

$$\nabla \cdot \vec{D} = \rho \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$  \hspace{1cm} (1)

$$\nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$  \hspace{1cm} (2)
where $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$. Here $\epsilon$ is a constant called the permittivity and $\mu$ a constant called the permeability, while $\rho$ and $\vec{J}$ are any free sources. Given the widespread application of these equations in electrical engineering among many other areas of applied science, and that these equations are in general too complex to be solved analytically, it is of great importance to be able to computationally simulate a system of interest. While a standard computational method already exists known as the Finite-Difference Time-Domain (FDTD) or Yee method, the algorithm has limitations: For one, the method involves having staggered grids for $\vec{E}$ and $\vec{H}$ in order to calculate the curls more easily, but these field grids must also be determined at all points in the computational domain at a time. For systems with a very fine geometry, this will have high computational demands. The method even performs poorly for simple systems such as a very long wire, where having to determine both computational grids results in computational power being used for largely redundant calculations.

3 Quantum Lattice Algorithms

The shortcomings of the FDTD method lead us to consider a new approach for computationally solving Maxwell’s equations. While many methods might be considered in this aim, here we consider quantum lattice algorithms. For a quantum algorithm, where one is performing a quantum computation on a set of qubits, called a register, it is required that all operations uphold the probabilistic interpretation of quantum mechanics. That is, it is required that $|\alpha(t)|^2 + |\beta(t)|^2 = 1$ for all time $t$ for a single qubit system. Thus any operator applied to the system, represented as a vector, must be unitary in order to uphold normality for the probabilistic interpretation.

With qubits in place of classical bits, a QLA represents a system over a lattice. By applying unitary operators to the state ket, the system is then evolved over the lattice. The unitary operators themselves are made up of an interleaved sequence of
non-commuting "collision" and "streaming" operators. The collision operators serve to entangle specific qubits together while the streaming operators move this entanglement throughout the lattice in discrete steps. For sufficiently small steps this process will perturbatively recover the behavior of the continuous Maxwell system. The QLA method outlined below can be used using classical bits, but with the potential to be implemented on a quantum computer, could offer even great computational benefits through quantum parallelism and entanglement when such technology exists.

3.1 Khan formulation of Maxwell’s Equations

In order to simulate Maxwell’s equations with a QLA, it is necessary for the Maxwell system be expressed in a form that can be acted on by the collide-stream operators. For a suitable representation one can utilize the work of Khan [1]. Here the Riemann Silberstein (RS) vector is introduced:

\[ \vec{F}^\pm = \frac{1}{\sqrt{2}}[\sqrt{\epsilon}\vec{E} \pm i\sqrt{\mu}\vec{B}] \]  

(3)

which defines the two polarizations, or field directions in essence, of the Maxwell system. The components of this are then used to define our qubit state vectors:

\[ \Psi^+ = \begin{pmatrix} -F_x^+ + iF_y^+ \\ F_z^+ \\ F_x^+ + iF_y^+ \end{pmatrix} ; \Psi^- = \begin{pmatrix} -F_x^- - iF_y^- \\ F_z^- \\ F_x^- - iF_y^- \end{pmatrix} \]  

(4)

Depending on the exact system, it can be sufficient to just use one of the two polarizations. However, in the more general form of an inhomogeneous medium, there is coupling between the two field polarizations so both are required. Using these four-spinors, Khan derives the following representation for an inhomogeneous Maxwell
system:

\[
\frac{\partial}{\partial t} \left( \begin{array}{c} \Psi^+ \\ \Psi^- \end{array} \right) - \frac{1}{2} \frac{\partial \ln v}{\partial t} \left( \begin{array}{c} \Psi^+ \\ \Psi^- \end{array} \right) + \frac{i}{2} M_\alpha \frac{\partial \ln h}{\partial t} \left( \begin{array}{c} \Psi^+ \\ \Psi^- \end{array} \right) = \frac{1}{2} \mu \left( \begin{array}{c} M \cdot \nabla + \Sigma \cdot \nabla v \\ -i \beta (\Sigma \cdot \nabla h) \alpha_y \end{array} \right) \left( \begin{array}{c} \Psi^+ \\ \Psi^- \end{array} \right) - \left( \begin{array}{c} W^+ \\ W^- \end{array} \right) \right)
\]

where \( v \) is the normalized phase velocity of electromagnetic wave in the medium and \( h \) is the normalized resistance, given respectively by:

\[
v = \frac{1}{\sqrt{\epsilon \mu}}, \quad h = \sqrt{\frac{\mu}{\epsilon}}
\]

Also from above, \( M = \sigma \cdot I_2 \) where \( \sigma = < \sigma_x, \sigma_y, \sigma_z > \) are the Pauli spin matrices and \( I_2 = < I_2, I_2, I_2 > \) where \( I_2 \) is the 2x2 identity matrix. The Pauli spin matrices are:

\[
\begin{align*}
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

Furthermore \( \alpha \) and \( \Sigma \) are given by

\[
\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}
\]

and \( W^\pm \) is the current and charge density matrix given by:

\[
W^\pm = \frac{1}{\sqrt{2\epsilon}} \begin{pmatrix} -J_x \pm iJ_y \\ J_z - \nu \rho \\ J_z + \nu \rho \\ J_x \pm iJ_y \end{pmatrix}
\]

5
It is important to note that eq. (5.) contains both Hermitian and anti-Hermitian operators, which will create some issues in creating unitary operators for our QLA, though there are various potential workarounds for this.

3.2 Collide-Stream Operators

With the field components now defined in a suitable matrix form, we proceed with defining the unitary operators that will perturb the system across the lattice in discrete time steps. For simplicity, we will consider only propagation in a single dimension, the x-component. Thus we use an 8-qubit system, one qubit per field component of Khan’s representation of Maxwell’s equations. Simplifying eq. (5) the inhomogeneous system, we end up with:

\[
\frac{\partial}{\partial t} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = -\frac{1}{n(x) \partial x} \begin{pmatrix} q_3 \\ q_0 \\ q_1 \\ q_2 \end{pmatrix} - \frac{n'(x)}{2n^2(x)} \begin{pmatrix} q_1 + q_6 \\ q_0 - q_7 \\ q_3 - q_4 \\ q_2 + q_5 \end{pmatrix}
\]

(10)

\[
\frac{\partial}{\partial t} \begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_7 \end{pmatrix} = -\frac{1}{n(x) \partial x} \begin{pmatrix} q_7 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} - \frac{n'(x)}{2n^2(x)} \begin{pmatrix} q_5 + q_2 \\ q_4 - q_3 \\ q_7 - q_0 \\ q_6 + q_1 \end{pmatrix}
\]

(11)

In this form it is clearly seen which qubits must be coupled by the operators in order to recover the dynamics of the continuous system. That is, for the \( \frac{\partial}{\partial x} \) term we must have \( q_0 \leftrightarrow q_2 \) and \( q_1 \leftrightarrow q_3 \) for (6), and \( q_4 \leftrightarrow q_6 \) and \( q_4 \leftrightarrow q_6 \).

Keeping in mind that we would like our operators to be unitary, we use the matrix
of the form \([2][3]\):

\[
C_x(\theta) = \begin{pmatrix}
\cos \theta & 0 & \sin \theta & 0 & 0 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & \sin \theta & 0 & 0 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 & 0 & 0 & 0 & 0 \\
0 & -\sin \theta & 0 & \cos \theta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \theta & 0 & \sin \theta & 0 \\
0 & 0 & 0 & 0 & 0 & \cos \theta & 0 & \sin \theta \\
0 & 0 & 0 & 0 & -\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sin \theta & \cos \theta
\end{pmatrix}
\] (12)

and its adjoint \(C^{\dagger}(\theta)\) for our collision operator. For our streaming operators, we will stream only half of the qubits at a time in order to couple the qubit collisions. Since our collision operator needs to couple \(q_0 \leftrightarrow q_2, q_1 \leftrightarrow q_3, q_4 \leftrightarrow q_6, q_4 \leftrightarrow q_6\) by (6) and (7), the first streaming operator \(S^{01}_{\pm x}\) will stream qubits \(q_0, q_1, q_4,\) and \(q_5\), and the second streaming operator \(S^{23}_{\pm x}\) will stream \(q_2, q_3, q_6,\) and \(q_7\). Thus the streaming operators act on the qubit vectors as \([2][3]\):

\[
S^{01}_{\pm x} \begin{pmatrix} q_0(x, t) \\ q_1(x, t) \\ q_2(x, t) \\ q_3(x, t) \\ q_4(x, t) \\ q_5(x, t) \\ q_6(x, t) \\ q_7(x, t) \end{pmatrix} = \begin{pmatrix} q_0(x \pm 1, t) \\ q_1(x \pm 1, t) \\ q_2(x, t) \\ q_3(x, t) \\ q_4(x \pm 1, t) \\ q_5(x \pm 1, t) \\ q_6(x, t) \\ q_7(x, t) \end{pmatrix}
\]

; \(S^{23}_{\pm x} \begin{pmatrix} q_0(x, t) \\ q_1(x, t) \\ q_2(x, t) \\ q_3(x, t) \\ q_4(x, t) \\ q_5(x, t) \\ q_6(x, t) \\ q_7(x, t) \end{pmatrix} = \begin{pmatrix} q_0(x, t) \\ q_1(x, t) \\ q_2(x, t) \\ q_3(x \pm 1, t) \\ q_4(x, t) \\ q_5(x, t) \\ q_6(x, t) \\ q_7(x \pm 1, t) \end{pmatrix}
\] (13)

With the collision and streaming operators defined we can now define the interleaved
sequence [2][3]:

\[
U_x = S^0_{-x} C_x S^0_{+x} C_x \dagger S^{23}_{-x} C_x S^{23}_{+x} C_x \dagger
\]  

(14)

\[
U_x^T = S^0_{+x} C_x \dagger S^0_{-x} C_x S^{23}_{+x} C_x \dagger S^{23}_{-x} C_x
\]  

(15)

Evaluating \( U^T U \) will give us the \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial x} \) terms where our angle is \( \theta = \frac{\epsilon}{4n(x)} \).

However, to recover the inhomogeneous term with the \( \frac{n'(x)}{2n^2(x)} \) term, we must introduce two more coupling operators. We note that for these terms \( q_0 \leftrightarrow q_1, q_2 \leftrightarrow q_3, q_4 \leftrightarrow q_5, q_6 \leftrightarrow q_7 \), so we introduce the operator [2][3]:

\[
V_1(\alpha) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \alpha & -\sin \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \alpha & -\sin \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cos \alpha & -\sin \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & -\sin \alpha & \cos \alpha
\end{pmatrix}
\]  

(16)

Likewise we have \( q_0 \leftrightarrow q_6, q_1 \leftrightarrow q_7, q_2 \leftrightarrow q_4, q_3 \leftrightarrow q_5 \), so we introduce [2][3]:

\[
V_2(\alpha) = \begin{pmatrix}
\cos \alpha & 0 & 0 & 0 & 0 & 0 & -\sin \alpha & 0 \\
0 & \cos \alpha & 0 & 0 & 0 & 0 & 0 & \sin \alpha \\
0 & 0 & \cos \alpha & 0 & \sin \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \alpha & 0 & -\sin \alpha & 0 & 0 \\
0 & 0 & -\sin \alpha & 0 & \cos \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & \sin \alpha & 0 & \cos \alpha & 0 & 0 \\
\sin \alpha & 0 & 0 & 0 & 0 & 0 & \cos \alpha & 0 \\
0 & -\sin \alpha & 0 & 0 & 0 & 0 & 0 & \cos \alpha
\end{pmatrix}
\]  

(17)
with the angle $\alpha = \epsilon \frac{n'(x)}{2n^2(x)}$. Here we note that $V_1(\alpha)$ is not unitary as a result of anti-hermitian matrices in eq. (5). Thus a hybrid quantum-classical approach might be required in order to fully encode this whole QLA onto a quantum computer. Another potential option would be to represent this final operator as a linear combination of unitary operators [5]. The final QLA then for the x-direction becomes:

$$q(x, t + \delta t) = V_2(\alpha)V_1(\alpha)U_x^T U_x q(t)$$

(18)

Applying this time-evolution operator recursively onto our register will evolve the system throughout the lattice, perturbatively recovering the system over the parameter $\epsilon$. This method offers the advantage over FTDT of not needing to worry about calculating staggered field grids over the whole computational domain. This algorithm can also be encoded on a classical computer, allowing utilization while quantum computing technology remains in its infancy.

### 4 Conclusion and Further Research

The above exposition’s goal was to explain the essential process and mathematical details behind a Quantum Lattice algorithm for Maxwell’s equations. Thus, only a basic x-propagation, inhomogeneous system was considered using the minimum 8-qubit algorithm required for Khan’s representation of Maxwell’s equations. Other areas to explore include the accuracy of the simulation process as well as explaining the process for propagation in different components and in higher dimensional systems. Connected to this, the relation between the structure of the different Pauli spin matrices and the propagation in a given direction could be explored further. That is, because of the diagonal entries in $\sigma_z$ for instance, a QLA for z-propagation requires twice as many qubits per lattice site (i.e. 16 qubit system) in order to produce the required coupling of qubits.
References


Algorithm for Maxwell’s Equations in Inhomogenous Media,
