Applications of Group Theory to Fundamental Particle Physics

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Abstract

Group theory plays a vital role in quantum field theory, which explains the behavior of the fundamental particles of nature to an extraordinary degree of precision. In particular, the existence of symmetry groups leads directly to the existence of conserved quantities. In addition, there are three gauge symmetries, U(1), SU(2), and SU(3) which explain the electromagnetic, weak, and strong forces and their mediating bosons.

I. Introduction

Quantum field theory has emerged in the past century as the most precise theory of nature ever developed, with experimental agreement better than one part in a billion for certain measurements. It explains the three fundamental forces of nature which play a measurable role in particle interactions: electromagnetism, the strong force, and the weak force. The electromagnetic force acts upon electrically charged particles and is mediated by the photon. The strong force acts upon quarks and gluons and is mediated by the gluon. The weak force acts upon all fermions and is mediated by the W and Z bosons.

A large part of this theory is based upon the recognition of the fundamental symmetries of a system, and describing these symmetries through the use of groups. Based on all the evidence available, it appears that the laws of nature are invariant under translations in time and space, rotations, and changes in velocity. This will then force our theory to be invariant under the transformations brought on by the Poincaré group. At the same time, we will show how Noether's Theorem will lead from these symmetries to physical invariants, such as angular momentum and energy. We will then speak more abstractly about invariances under phase factors, which will form the heart of electrodynamics and the existence of some conserved quantity, which we will identify as electric charge. Similar symmetries will give us the strong and weak forces, the latter of which we will be able to merge with electromagnetism to obtain the electroweak force.

II. Poincaré Group

The Poincaré group encompasses invariance of our theory under spatial rotations, speed boosts, and spatial and temporal translations. We will therefore build this group by adding these components one at a time.

II..1 Rotation Group

As far as we can tell, the laws of nature do not have any preferred direction, so we are free to work in rotated reference frames. Therefore, we would like for our theories to behave the same if we rotate our coordinates. Let x be some position vector. Then, rotations in three dimensions may be represented as a three by three matrix. We may ask what properties this matrix should have. First, we would like our rotations to preserve lengths. In particular, if we have a rotation R and pick one of its eigenvectors, call it y with the associated eigenvalue λ, it must be that $|Ry| = |\lambda y| = |y|$. This means that R must have all its eigenvalues have unit magnitude, so, since the rotation matrix should physically have only real entries, its determinant will be ± 1. The negative determinent represents a transformation of the system to its mirror image, a so-called parity transformation. However, certain aspects of physics, namely, the weak force, violate parity and behave differently under such a transformation. Although this is a very interesting topic, we will not mention it further here. The requirement that distance be preserved also places the more stringent requirement that the matrix be orthogonal. Adopting the common notation, we will call this set of rotations SO(3) - the special orthogonal group in three dimensions. We may easily verify that this will indeed be a group under matrix multiplication. It is closed on physical grounds, since combining two rotations will form a new rotation. Associativity is given by the known properties of matrix multiplication. The identity exists as the unit matrix, and corresponds to not touching the sytem at all. Finally, each element will have an inverse: this may be seen both physically, since we can always rotate a body back to recover its original position, and mathematically, since the determinent of these matrices is one. We are therefore justified in calling this the special orthogonal *group*.

II..2 Lorentz Group

We will now add in speed boosts, which are just shifts in the velocity of our reference frame. Einsteinian relativity is based upon the fact that our laws of physics should not change if we shift the velocity of our coordinates.* If we let x be a four vector, i.e., a spatial vector plus one coordinate representing time, we will define Lorentz transformations as those transformations preserving the invariant interval $s^2 = t^2 - (x^2 + y^2 + z^2)$, which is known to be preserved under boosts and rotations in special relativity. We also require that there be no changes in parity, so that the determinent is one. This leads us to the Lorentz group. We may realize the elements of this group as 4-by-4 matrices, typically denoted by Λ, as follows: suppose that v is a four-vector. Then, our invariant interval may be written as $v^T D v$, where D is the matrix having 1 in the upper left corner, -1 on the rest of the diagonal, and 0 elsewhere. If we want our interval to be preserved under Lorentz transformations, it must be that $v^T D v = (\Lambda v)^T D \Lambda v = v^T \Lambda^T D \Lambda v$. If this is to be true for all four-vectors v, we must have that $\Lambda^T D \Lambda = D$. (Otherwise, we could take the matrix $\Lambda^T D \Lambda - D$ and find some non-zero eigenvalue λ and corresponding eigenvector w, so that $(\Lambda w)^T D \Lambda w - w^T D w = w^T(\Lambda^T D \Lambda - D)w = w^T \lambda w \neq 0$.) It is also clear that any matrix A such that $A^T D A = D$ will preserve the invariant interval, so that the general criterion for a 4-by-4 matrix to represent a Lorentz transform is that $\Lambda^T D \Lambda = D$ and $\det(\Lambda) = 1$. It is easily seen that these 4-by-4 matrices will form a group. If $|\Lambda_1 x| = |x|$ and $|\Lambda_2 y| = |y|$ for all four vectors x, y, then, setting $y = \Lambda_1 x$, we get $|\Lambda_2 \Lambda_1 x| = |\Lambda_1 x| = |x|$, so the matrix product will also preserve the invariant length. We also see that the determinent of the product of two such matrices will be one, since the determinent of a product is the product of determinents. Therefore, the set is

*In fact, Newtonian mechanics also contains this same invariance. The difference is that Einstein identifies the constant speed of light as one of the invariants, which has a fundamental impact on the way in which we must carry out our transformations.

closed under matrix multiplication. We see associativity by the laws of matrix multiplication. We see that the identity matrix will be the four by four identity matrix, and, since each matrix has a determinent of one, it is invertible. Therefore, they form a group.

II..3 Poincaré Group

We now add in spatial and temporal translations. Mathematically, this is equivalent to adding some four vector Δx to our original four vector after applying the Lorentz transformation. So, we may write such a transformation acting on a four-vector v as $\Lambda v + \Delta x$. We can simplify the notation if we make an affine transform so that the four-vector v is mapped to a vector with five components, the first four are identical to those of v, and the fifth being 1. For instance,

$$\begin{bmatrix} t_0 \\ x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$

Then, our Poincaré transform is a five-by-five matrix with a Lorentz transform Λ making up the upper 4-by-4 section, the lower right corner being 1, the rest of the rightmost column being our four-vector Δx, and the four slots below Λ being filled with zeros. Such a transformation will look like

$$\begin{bmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} & \Lambda_{03} & \delta t \\ \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \delta x \\ \Lambda_{20} & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \delta y \\ \Lambda_{30} & \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \delta z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} \Lambda & \Delta x \\ ----- & ----- \\ 0 & 1 \end{array}\right]$$

where Λ_{ij} are the components of a Lorentz transform and $\delta t, \delta x, \delta y, \delta z$ are the components of the four-vector Δx above. We see that these transformations will form a group. Since we have seen that the Lorentz transforms form a group, writing the partitioned matrices,

$$\left[\begin{array}{c|c} \Lambda_1 & \Delta x_1 \\ ----- & ----- \\ 0 & 1 \end{array}\right] \times \left[\begin{array}{c|c} \Lambda_2 & \Delta x_2 \\ ----- & ----- \\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \Lambda_1 \Lambda_2 & \Lambda_1 \Delta x_2 + \Delta x_1 \\ ----- & ----- \\ 0 & 1 \end{array}\right]$$

will be a new Poincaré transform. Then, our set of Poincaré transformations is closed under composition. We see that it is associative based on the rules of matrix multiplication. There is an identity, namely I_5, and, each element has an inverse, namely

$$\left[\begin{array}{c|c} \Lambda^{-1} & -\Lambda^{-1} \Delta x \\ ----- & ----- \\ 0 & 1 \end{array}\right]$$

Therefore, these transformations form a group.

II..4 Noether's Theorem

Up until now, the point of these symmetries has been very abstract. In fact, we have not even made special use of the fact that these are symmetries of nature. We now will explain that matter.

If some coordinate transformation is a symmetry, then it should not matter whether we use our old or new coordinates. For instance, if $\Psi(x)$ is some wavefunction, then we may transform it as $\Psi'(x) = \Psi(Tx)$, where T is some transformation, for example a Poincaré transformation. We expect that Ψ' should in some way depend on Ψ, so that we may write $\Psi' = U\Psi$, where U is some as-yet undetermined operator. This should not change the physics, so that $\langle\Psi|\Psi\rangle = \langle\Psi'|\Psi'\rangle$.[†] Then, $\langle U\Psi|U\Psi\rangle = \langle\Psi|\Psi\rangle$. But this implies that $\langle\Psi|U^\dagger U\Psi\rangle = \langle\Psi|\Psi\rangle$.[‡] We see here that it is sufficient to require that $U^\dagger U = I$, the identity. We also see that it must be necessary, since, if $U^\dagger U \neq I$, we could pick out some small region of space where the two differed and take the integral of our wavefunction over that region (representing the probability of finding our particle there). The two integrals would give different answers, so that the transformation would be changing the physics, making it not a symmetry. We call any operator with the properties of U a unitary operator. We also require energies to be the same. Taking the expectation value of the energy, $\langle\Psi|H|\Psi\rangle = \langle U\Psi|H|U\Psi\rangle$, we get that $\langle\Psi|H|\Psi\rangle = \langle\Psi|U^\dagger HU|\Psi\rangle$, implying that $H = U^\dagger HU$. Since $U^\dagger = U^{-1}$, we immediately see that $UH = HU$. Therefore, $[H, U] = HU - UH = 0$, so that our unitary operator commutes with the Hamiltonian. Any unitary operator may be written as e^{iA}, where A is a Hermitian operator ($A = A^\dagger$).[§] Then, if we make some small change to our wavefunction by operating on it with $e^{i\epsilon A}$, with $\epsilon << 1$, we have that $U \approx I + i\epsilon A$. Then $[H, I + i\epsilon A] = 0$, so that $H + i\epsilon HA = H + i\epsilon AH$. Therefore, $HA = AH$, so A also commutes with the Hamiltonian. But there is a theorem in quantum mechanics that any operator which commutes with the Hamiltonian is an invariant.[¶] Since U also commutes with the Hamiltonian, it would seem strange that we invented this A to do the job. But, A is Hermitian, while U is (in general) not, so A will correspond to some physically real thing (energy, momentum, etc.). Therefore, we have some real, invariant quantity from our symmetry. This is Noether's Theorem.[‖]

III. Gauge Bosons

In the standard model, there are fermions, which have half-integer spin, and the bosons, which have integer spin. The fermions are broadly split into two categories: the quarks, which interact by the strong force and are bound together in hadrons, and the leptons, which do not interact by the strong force. The quarks each have a flavor (up, down, charm, strange, top, or bottom) and some color charge (red, green, or blue). The leptons are further subdivided into the neutrinos, which are electrically neutral, and the charged leptons (the electron, muon, and tau) which have identical electric charges. The gauge bosons, which include the photons, W/Z bosons, and gluons, mediate the three[**] fundamental forces between the fermions.

III..1 Electromagnetic Interaction

Let's seek out some invariants. The easiest way to transform the wavefunction in an invariant way is to take $\Psi \to e^{iq}\Psi$. Of course, $\langle e^{iq}\Psi|e^{iq}\Psi\rangle = \langle\Psi|e^{-iq}e^{iq}\Psi\rangle = \langle\Psi|\Psi\rangle$, so the transformation is unitary. Since this transformation is just mutiplication by a scalar, it will commute with any

[†] This refers to taking the integral of $\Psi^*\Psi$ over all space and represents the probability of finding the particle somewhere, so it is identically one. The arguments have been suppressed.

[‡] $U^\dagger$ is the Hermitian conjugate of U, defined such that $\langle Ux|y\rangle = \langle x|U^\dagger y\rangle$ for all wavefunctions x, y.

[§] The meaning of an operator as an exponent is to just take the Taylor series: $e^B = I + B + B^2/2! + \ldots$

[¶] In general, the change of the measured value of an operator with time is given by $\frac{d}{dt}\langle A\rangle = i/\hbar[H, A]$.

[‖] This proof based on the lectures of Mark Thomson.

[**] Gravity is excluded, since it is too weak to play a role in any experiments we are able to do in particle physics. Its inclusion is the hope of many theoretical physicists.

observable. We easily see that, for small values of q, $e^{iq} = I + iq$, so our invariant will be the scalar q. In fact, this may be identified with electric charge, so we see that conservation of charge follows simply from the symmetry of our wavefunction under a change of phase. It is trivial to show that the set of scalars written in the form e^{iq} form a group for real q, and this is called the $U(1)$ group. The fact that this group has a single generator implies that it is mediated by a single gauge boson, the photon.

III..2 Strong Interaction

The quarks are known to have three colors: red, green, and blue.[††] It turns out that we can rotate the quark colors as we please without changing anything fundamental about the physics. If we imagine the color content of a quark as a column vector

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

with the R, G, B representing the color charge of the quark for each color, we may imagine some roation operator in color space. As we had seen above, it must be unitary. But any unitary matrix may be written as e^{iH} for some Hermitian matrix H. A three by three Hermitian matrix will have the generic form

$$\begin{bmatrix} a & b+ci & d+ei \\ b-ci & f & g+hi \\ d-ei & g-hi & k \end{bmatrix}$$

where $a-k$ are 9 independent real parameters. However, we will remove from consideration the unitary matrix corresponding to just an overall phase factor. This is formed as e^{iI}, where I is the unit matrix, so we are left with just 8 independent Hermitian generators. Six of these will be formed by taking our generic Hermitian matrix and setting to zero all diagonal terms and all but one of b, c, d, e, g, h, which is set to one. For instance, picking $d = 1$, we get

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The other two are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{bmatrix}$$

The unitary matrices generated by these eight Hermitian matrices form the SU(3) symmetry group. (This is a group under multiplication, since the eight generating matrices and their linear combinations form a group under addition, and the two are related by $U = e^{iH}$. From here, making the normal group checks becomes trivial.) Since there are eight generating matrices, we have eight

[††]These are merely labels for some sort of exotic charge and have no relation to those colors which we are able to see.

different gluons, with their coupling to the different quarks described by the nonzero elements in their associated matrix. For instance, the gluons described by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

will be involved in interchanging the colors of interacting red and blue quarks. The fact that the gluons themselves have this color charge means that they are also subject to strong force interactions.

III..3 Weak Interaction

We may find a new type of symmetry if we consider transformations between different flavors of quarks (such as up and down quarks) or between neutrinos and their associated charged leptons (for instance, an electron and electron neutrino).[‡‡] As before, we may define some two by two unitary tranformation matrix, which will be a symmetry of the system. Again, these unitary matrices may be written as e^{iH} for Hermitian H, and, similar to above, these H's take the form

$$\begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix}$$

As above, we discard the identity matrix, since that leaves us with just a phase change. We then have three generators:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Together, the unitary rotations formed by these matrices form the SU(2) symmetry group. The first two of these matrices correspond to the $W^{\pm}$ bosons, which mediate charged-current weak interactions (where the charges of the interacting particles change, such as an electron becoming an electron neutrino). The third matrix and the generator of the U(1) symmetry group discussed above actually mix, so that linear combinations of those will correspond to the photon and the Z boson, which mediates neutral-current weak interactions (where the interacting particles do not change their charge). This is referred to as electroweak unification.

[‡‡]Properly, we must specify "left-handed" leptons, which the weak force treats differently than those which are right handed. Handedness, at high velocities, relates to how the spin of a particle aligns with its direction of motion, but the full story is more complicated.

IV. Future Work

Despite its experimental successes, there are still a number of known issues with the standard model. Most obvious is that it does not include gravity. Although, gravity is not expected to affect particle interactions which we can experimentally probe in the forseeable future, the fact that it cannot be explained in the same language as the other forces requires that there be some more complete theory, and there are a fair number of theoretical attempts to explain quantum gravity.

On a more reasonable timescale, there are some strange coincidences in the standard model: for instance, the fact that the mass of the Higgs boson (125 GeV/c^2) is relatively small compared to the Planck mass ($\sim 10^{19}$ GeV/c^2) implies that certain terms in the physics calculations cancel across many orders of magnitude. These cancellations may be achieved naturally if we include a whole new set of particles such that each standard model particle has a supersymmetric partner. There is currently a large amount of experimental effort searching for these particles, with both the ATLAS and CMS collaborations at CERN's LHC hoping to find evidence for these particles. Additionally, there are attempts to fit all of this into some sort of overarching symmetry group, but, as yet, nothing suitable has been found.

V. Conclusion

We then see that symmetries and the associated symmetry groups have a profound impact on particle physics. On the one hand, these symmetries are responsible for the existence of conserved quantities. On the other hand, certain gauge symmetries lead to the existence of gauge bosons which mediate the fundamental forces of nature. The electromagnetic and weak forces have been shown to be two aspects of the same overarching electroweak force, and unification of the rest of the forces is of great theoretical interest.

VI. Acknowledgements

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