Problem Set I.1

1 Give an example where a combination of three nonzero vectors in $\mathbf{R}^4$ is the zero vector. Then write your example in the form $A\boldsymbol{x} = \mathbf{0}$. What are the shapes of A and $\boldsymbol{x}$ and $\mathbf{0}$?

2 Suppose a combination of the columns of A equals a different combination of those columns. Write that as $A\boldsymbol{x} = A\boldsymbol{y}$. Find two combinations of the columns of A that equal the zero vector (in matrix language, find two solutions to $A\boldsymbol{z} = \mathbf{0}$).

3 (Practice with subscripts) The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are in m-dimensional space $\mathbf{R}^m$, and a combination $c_1\boldsymbol{a}_1 + \cdots + c_n\boldsymbol{a}_n$ is the zero vector. That statement is at the vector level.

(1) Write that statement at the matrix level. Use the matrix A with the $\boldsymbol{a}$'s in its columns and use the column vector $\boldsymbol{c} = (c_1, \ldots, c_n)$.

(2) Write that statement at the scalar level. Use subscripts and sigma notation to add up numbers. The column vector $\boldsymbol{a}_j$ has components $a_{1j}, a_{2j}, \ldots, a_{mj}$.

4 Suppose A is the 3 by 3 matrix **ones**(3, 3) of all ones. Find two independent vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ that solve $A\boldsymbol{x} = \mathbf{0}$ and $A\boldsymbol{y} = \mathbf{0}$. Write that first equation $A\boldsymbol{x} = \mathbf{0}$ (with numbers) as a combination of the columns of A. Why don't I ask for a third independent vector with $A\boldsymbol{z} = \mathbf{0}$?

5 The linear combinations of $\boldsymbol{v} = (1,1,0)$ and $\boldsymbol{w} = (0,1,1)$ fill a plane in $\mathbf{R}^3$.

(a) Find a vector $\boldsymbol{z}$ that is perpendicular to $\boldsymbol{v}$ and $\boldsymbol{w}$. Then $\boldsymbol{z}$ is perpendicular to every vector $c\boldsymbol{v} + d\boldsymbol{w}$ on the plane: $(c\boldsymbol{v} + d\boldsymbol{w})^{\mathrm{T}}\boldsymbol{z} = c\boldsymbol{v}^{\mathrm{T}}\boldsymbol{z} + d\boldsymbol{w}^{\mathrm{T}}\boldsymbol{z} = 0 + 0$.

(b) Find a vector $\boldsymbol{u}$ that is not on the plane. Check that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{z} \neq 0$.

6 If three corners of a parallelogram are $(1,1)$, $(4,2)$, and $(1,3)$, what are all three of the possible fourth corners? Draw two of them.

7 Describe the column space of $A = [\,\boldsymbol{v} \;\; \boldsymbol{w} \;\; \boldsymbol{v} + 2\boldsymbol{w}\,]$. Describe the nullspace of A: all vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ that solve $A\boldsymbol{x} = \mathbf{0}$. Add the "dimensions" of that plane (the column space of A) and that line (the nullspace of A):

dimension of column space + dimension of nullspace = number of columns

8 $A = CR$ is a representation of the columns of A in the basis formed by the columns of C with coefficients in R. If $A_{ij} = j^2$ is 3 by 3, write down A and C and R.

9 Suppose the column space of an m by n matrix is all of $\mathbf{R}^3$. What can you say about m? What can you say about n? What can you say about the rank r?

10 Find the matrices C_1 and C_2 containing independent columns of A_1 and A_2:

$$A_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

11 Factor each of those matrices into $A = CR$. The matrix R will contain the numbers that multiply columns of C to recover columns of A.

This is one way to look at matrix multiplication: **C times each column of R.**

12 Produce a basis for the column spaces of A_1 and A_2. What are the *dimensions* of those column spaces—the number of independent vectors? What are the *ranks* of A_1 and A_2? How many independent rows in A_1 and A_2?

13 Create a 4 by 4 matrix A of rank 2. What shapes are C and R?

14 Suppose two matrices A and B have the same column space.

(a) Show that their row spaces can be different.

(b) Show that the matrices C (basic columns) can be different.

(c) What number will be the same for A and B?

15 If $A = CR$, the first row of A is a combination of the rows of R. Which part of which matrix holds the coefficients in that combination—the numbers that multiply the rows of R to produce row 1 of A?

16 The rows of R are a basis for the row space of A. *What does that sentence mean*?

17 For these matrices with square blocks, find $A = CR$. What ranks?

$$A_1 = \begin{bmatrix} \text{zeros} & \text{ones} \\ \text{ones} & \text{ones} \end{bmatrix}_{4 \times 4} \qquad A_2 = \begin{bmatrix} A_1 \\ A_1 \end{bmatrix}_{8 \times 4} \qquad A_3 = \begin{bmatrix} A_1 & A_1 \\ A_1 & A_1 \end{bmatrix}_{8 \times 8}$$

18 If $A = CR$, what are the CR factors of the matrix $\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$?

19 "Elimination" subtracts a number ℓ_{ij} times row j from row i: a "row operation." Show how those steps can reduce the matrix A in Example 4 to R (except that this row echelon form R has a row of zeros). The rank won't change!

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} \;\rightarrow\; \rightarrow \quad R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \textbf{rref}(A).$$

This page is about the factorization $A = CR$ and its close relative $A = CMR$. As before, C has r independent columns taken from A. *The new matrix* $\boldsymbol{R}$ *has* r *independent rows, also taken directly from* A. The r by r **"mixing matrix"** is $\boldsymbol{M}$. This invertible matrix makes $A = CMR$ a true equation.

The rows of R (not bold) were chosen to produce $A = CR$, but those rows of *R did not come directly from A.* We will see that R has the form MR (bold R).

$$\textbf{Rank-1 example} \quad A = CR = CMR \qquad \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix}$$

In this case M is just 1 by 1. How do we find M in other examples of $A = CMR$? C and R are not square. They have *one-sided* inverses. *We invert* C^TC and RR^T.

$$\boxed{A = CMR} \quad C^TAR^T = C^TC\,M\,RR^T \quad \boxed{M = (C^TC)^{-1}(C^TAR^T)(RR^T)^{-1}} \quad (*)$$

Here are extra problems to give practice with all these rectangular matrices of rank r. C^TC and RR^T have rank r so they are invertible (see the last page of Section I.3).

20 Show that equation $(*)$ produces $M = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$ in the small example above.

21 The rank-2 example in the text produced $A = CR$ in equation (2):

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = CR$$

Choose rows 1 and 2 directly from A to go into $\boldsymbol{R}$. Then from equation $(*)$, find the 2 by 2 matrix M that produces $A = CMR$. Fractions enter the inverse of matrices:

$$\textbf{Inverse of a 2 by 2 matrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (**)$$

22 Show that this formula $(**)$ breaks down if $\begin{bmatrix} b \\ d \end{bmatrix} = m \begin{bmatrix} a \\ c \end{bmatrix}$: *dependent columns.*

23 Create a 3 by 2 matrix A with rank 1. Factor A into $A = CR$ and $A = CMR$.

24 Create a 3 by 2 matrix A with rank 2. Factor A into $A = CMR$.

The reason for this page is that the factorizations $A = CR$ and $A = CMR$ have jumped forward in importance for large matrices. When C takes columns directly from A, and R takes rows directly from A, those matrices preserve properties that are lost in the more famous QR and SVD factorizations. Where $A = QR$ and $A = U\Sigma V^T$ involve orthogonalizing the vectors, C and R keep the original data:

If A is nonnegative, so are C and R. If A is sparse, so are C and R.

I.2 Matrix-Matrix Multiplication AB

Inner products (*rows times columns*) produce each of the numbers in $AB = C$:

$$\begin{matrix} \textbf{row 2 of } A \\ \textbf{column 3 of } B \\ \textbf{give } c_{23} \textbf{ in } C \end{matrix} \qquad \begin{bmatrix} \cdot & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & b_{13} \\ \cdot & \cdot & b_{23} \\ \cdot & \cdot & b_{33} \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & c_{23} \\ \cdot & \cdot & \cdot \end{bmatrix} \qquad (1)$$

That dot product c_{23} = (row 2 of A) $\cdot$ (column 3 of B) is a sum of a's times b's:

$$c_{23} = a_{21}\,b_{13} + a_{22}\,b_{23} + a_{23}\,b_{33} = \sum_{k=1}^{3} a_{2k}\,b_{k3} \quad \text{and} \quad c_{ij} = \sum_{k=1}^{n} a_{ik}\,b_{kj}. \qquad (2)$$

This is how we usually compute each number in $AB = C$. But there is another way.

The other way to multiply AB is **columns of A times rows of B**. We need to see this! I start with numbers to make two key points: *one column* $\boldsymbol{u}$ times *one row* $\boldsymbol{v}^T$ produces a *matrix*. Concentrate first on that piece of AB. This matrix $\boldsymbol{uv}^T$ is especially simple:

$$\textbf{"Outer product"} \quad \boldsymbol{uv}^T = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 12 \\ 6 & 8 & 12 \\ 3 & 4 & 6 \end{bmatrix} = \textbf{"rank one matrix"}$$

An m by 1 matrix (a column $\boldsymbol{u}$) times a 1 by p matrix (a row $\boldsymbol{v}^T$) gives an m by p matrix. Notice what is special about the rank one matrix $\boldsymbol{uv}^T$:

$$\text{All columns of } \boldsymbol{uv}^T \text{ are multiples of } \boldsymbol{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \text{All rows are multiples of } \boldsymbol{v}^T = \begin{bmatrix} 3 & 4 & 6 \end{bmatrix}$$

The column space of $\boldsymbol{uv}^T$ is one-dimensional: *the line in the direction of* $\boldsymbol{u}$. The dimension of the column space (the number of independent columns) is the **rank of the matrix—a key number. All nonzero matrices $\boldsymbol{uv}^T$ have rank one.** They are the perfect building blocks for every matrix.

Notice also: **The row space of $\boldsymbol{uv}^T$ is the line through $\boldsymbol{v}$.** By definition, the row space of any matrix A is the column space $\mathbf{C}(A^T)$ of its transpose A^T. That way we stay with column vectors. In the example, we transpose $\boldsymbol{uv}^T$ **(exchange rows with columns)** to get the matrix $\boldsymbol{vu}^T$:

$$(\boldsymbol{uv}^T)^T = \begin{bmatrix} 6 & 8 & 12 \\ 6 & 8 & 12 \\ 3 & 4 & 6 \end{bmatrix}^T = \begin{bmatrix} 6 & 6 & 3 \\ 8 & 8 & 4 \\ 12 & 12 & 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} = \boldsymbol{vu}^T.$$

The diagonal matrix Λ contains real eigenvalues λ_1 to λ_n. Every real symmetric matrix S has n orthonormal eigenvectors $\boldsymbol{q}_1$ to $\boldsymbol{q}_n$. When multiplied by S, the eigenvectors keep the same direction. They are just rescaled by the number λ:

$$\textbf{Eigenvector } \boldsymbol{q} \textbf{ and eigenvalue } \lambda \qquad S\boldsymbol{q} = \lambda\boldsymbol{q} \tag{5}$$

Finding λ and $\boldsymbol{q}$ is not easy for a big matrix. But n pairs always exist when S is symmetric. Our purpose here is to see how $SQ = Q\Lambda$ comes column by column from $S\boldsymbol{q} = \lambda\boldsymbol{q}$:

$$SQ = S\begin{bmatrix} \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\boldsymbol{q}_1 & \cdots & \lambda_n\boldsymbol{q}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = Q\Lambda \tag{6}$$

Multiply $SQ = Q\Lambda$ by $Q^{-1} = Q^{\mathrm{T}}$ to get $S = Q\Lambda Q^{\mathrm{T}}$ = a symmetric matrix. Each eigenvalue λ_k and each eigenvector $\boldsymbol{q}_k$ contribute a rank one piece $\lambda_k\boldsymbol{q}_k\boldsymbol{q}_k^{\mathrm{T}}$ to S.

$$\textbf{Rank one pieces} \qquad S = (Q\Lambda)Q^{\mathrm{T}} = (\lambda_1\boldsymbol{q}_1)\boldsymbol{q}_1^{\mathrm{T}} + (\lambda_2\boldsymbol{q}_2)\boldsymbol{q}_2^{\mathrm{T}} + \cdots + (\lambda_n\boldsymbol{q}_n)\boldsymbol{q}_n^{\mathrm{T}} \tag{7}$$

$$\textbf{All symmetric} \qquad \text{The transpose of } \boldsymbol{q}_i\boldsymbol{q}_i^{\mathrm{T}} \text{ is } \boldsymbol{q}_i\boldsymbol{q}_i^{\mathrm{T}} \tag{8}$$

Please notice that the columns of $Q\Lambda$ are $\lambda_1\boldsymbol{q}_1$ to $\lambda_n\boldsymbol{q}_n$. When you multiply a matrix on the right by the diagonal matrix Λ, you multiply its *columns* by the λ's.

We close with a comment on the proof of this **Spectral Theorem** $S = Q\Lambda Q^{\mathrm{T}}$: Every symmetric S has n real eigenvalues and n orthonormal eigenvectors. Section I.6 will construct the eigenvalues as the roots of the nth degree polynomial $P_n(\lambda)$ = determinant of $S - \lambda I$. They are real numbers when $S = S^{\mathrm{T}}$. The delicate part of the proof comes when an eigenvalue λ_i is *repeated*—it is a double root or an Mth root from a factor $(\lambda - \lambda_j)^M$. In this case we need to produce M independent eigenvectors. The rank of $S - \lambda_j I$ must be $n - M$. This is true when $S = S^{\mathrm{T}}$. But it requires a proof.

Similarly the Singular Value Decomposition $A = U\Sigma V^{\mathrm{T}}$ requires extra patience when a singular value σ is repeated M times in the diagonal matrix Σ. Again there are M pairs of singular vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ with $A\boldsymbol{v} = \sigma\boldsymbol{u}$. Again this true statement requires proof.

Notation for rows We introduced the symbols $\boldsymbol{b}_1^*, \ldots, \boldsymbol{b}_n^*$ for the rows of the second matrix in AB. You might have expected $\boldsymbol{b}_1^{\mathrm{T}}, \ldots, \boldsymbol{b}_n^{\mathrm{T}}$ and that was our original choice. But this notation is not entirely clear—it seems to mean the transposes of the columns of B. Since that right hand factor could be U or R or Q^{T} or X^{-1} or V^{T}, it is safer to say definitely: *we want the rows of that matrix.*

G. Strang, *Multiplying and factoring matrices*, Amer. Math. Monthly **125** (2018) 223-230.

G. Strang, *Introduction to Linear Algebra*, 5th ed., Wellesley-Cambridge Press (2016).

Problem Set I.2

1 Suppose $A\boldsymbol{x} = \boldsymbol{0}$ and $A\boldsymbol{y} = \boldsymbol{0}$ (where $\boldsymbol{x}$ and $\boldsymbol{y}$ and $\boldsymbol{0}$ are vectors). Put those two statements together into one matrix equation $AB = C$. What are those matrices B and C? If the matrix A is m by n, what are the shapes of B and C?

2 Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are column vectors with components $a_1, \ldots, a_m$ and $b_1, \ldots, b_p$. Can you multiply $\boldsymbol{a}$ times $\boldsymbol{b}^{\mathrm{T}}$ (yes or no)? What is the shape of the answer $\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}$? What number is in row i, column j of $\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}$? What can you say about $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}$?

3 (Extension of Problem 2: Practice with subscripts) Instead of that one vector $\boldsymbol{a}$, suppose you have n vectors $\boldsymbol{a}_1$ to $\boldsymbol{a}_n$ in the columns of A. Suppose you have n vectors $\boldsymbol{b}_1^{\mathrm{T}}, \ldots, \boldsymbol{b}_n^{\mathrm{T}}$ in the rows of B.

(a) Give a "sum of rank one" formula for the matrix-matrix product AB.

(b) Give a formula for the i, j entry of that matrix-matrix product AB. Use sigma notation to add the i, j entries of each matrix $\boldsymbol{a}_k\boldsymbol{b}_k^{\mathrm{T}}$, found in Problem 2.

4 Suppose B has only one column ($p = 1$). So each row of B just has one number. A has columns $\boldsymbol{a}_1$ to $\boldsymbol{a}_n$ as usual. Write down the column times row formula for AB. In words, the m by 1 column vector AB is a combination of the _____.

5 Start with a matrix B. If we want to take combinations of its rows, we premultiply by A to get AB. If we want to take combinations of its columns, we postmultiply by C to get BC. For this question we will do both.

Row operations then column operations First AB then $(AB)C$

Column operations then row operations First BC then $A(BC)$

The **associative law** says that we get the same final result both ways.

Verify $(AB)C = A(BC)$ for $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$.

6 If A has columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ and $B = I$ is the identity matrix, what are the rank one matrices $\boldsymbol{a}_1\boldsymbol{b}_1^*$ and $\boldsymbol{a}_2\boldsymbol{b}_2^*$ and $\boldsymbol{a}_3\boldsymbol{b}_3^*$? They should add to $AI = A$.

7 *Fact*: The columns of AB are combinations of the columns of A. Then the column *space* of AB is *contained in* the column space of A. Give an example of A and B for which AB has a smaller column space than A.

8 To compute $C = AB = (m \text{ by } n)\,(n \text{ by } p)$, what order of the same three commands leads to columns times rows (outer products)?

Rows times columns	**Columns times rows**
For $i = 1$ to m	For...
For $j = 1$ to p	For...
For $k = 1$ to n	For...
$C(i,j) = C(i,j) + A(i,k) * B(k,j)$	$C =$

Problem Set I.3

1 Show that the nullspace of AB contains the nullspace of B. **If $Bx = 0$ then...**

2 Find a square matrix with rank $(A^2) <$ rank (A). Confirm that rank $(A^\mathrm{T}A) =$ rank (A).

3 How is the nullspace of C related to the nullspaces of A and B, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

4 If row space of A = column space of A, and also $\mathrm{N}(A) = \mathrm{N}(A^\mathrm{T})$, is A symmetric?

5 Four possibilities for the rank r and size m, n match four possibilities for $Ax = b$. Find four matrices A_1 to A_4 that show those possibilities:

$r = m = n$	$A_1x = b$ **has 1 solution for every b**
$r = m < n$	$A_2x = b$ **has 1 or ∞ solutions**
$r = n < m$	$A_3x = b$ **has 0 or 1 solution**
$r < m,\ r < n$	$A_4x = b$ **has 0 or ∞ solutions**

6 *(Important)* Show that **$A^\mathrm{T}A$ has the same nullspace as A**. Here is one approach: First, if Ax equals zero then $A^\mathrm{T}Ax$ equals ____. This proves $\mathrm{N}(A) \subset \mathrm{N}(A^\mathrm{T}A)$. Second, if $A^\mathrm{T}Ax = 0$ then $x^\mathrm{T}A^\mathrm{T}Ax = ||Ax||^2 = 0$. Deduce $\mathrm{N}(A^\mathrm{T}A) = \mathrm{N}(A)$.

7 Do A^2 and A always have the same nullspace? A is a square matrix.

8 Find the column space $\mathbf{C}(A)$ and the nullspace $\mathrm{N}(A)$ of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Remember that those are vector spaces, not just single vectors. This is an unusual example with $\mathbf{C}(A) = \mathrm{N}(A)$. It could not happen that $\mathbf{C}(A) = \mathrm{N}(A^\mathrm{T})$ because those two subspaces are orthogonal.

9 Draw a square and connect its corners to the center point: 5 nodes and 8 edges. Find the 8 by 5 incidence matrix A of this graph (rank $r = 5 - 1 = 4$). Find a vector x in $\mathrm{N}(A)$ and $8 - 4$ independent vectors y in $\mathrm{N}(A^\mathrm{T})$.

10 If $\mathrm{N}(A)$ is the zero vector, what vectors are in the nullspace of $B = [A\ A\ A]$?

11 For subspaces $\mathbf{S}$ and $\mathbf{T}$ of $\mathbf{R}^{10}$ with dimensions 2 and 7, what are all the possible dimensions of

(i) $\mathbf{S} \cap \mathbf{T}$ = {all vectors that are in both subspaces}

(ii) $\mathbf{S} + \mathbf{T}$ = {all sums $s + t$ with s in $\mathbf{S}$ and t in $\mathbf{T}$}

(iii) $\mathbf{S}^\perp$ = {all vectors in $\mathbf{R}^{10}$ that are perpendicular to every vector in $\mathbf{S}$}.

I.4 Elimination and $A = LU$

The first and most fundamental problem of linear algebra is to solve $Ax = b$. We are given the n by n matrix A and the n by 1 column vector b. We look for the solution vector x. Its components $x_1, x_2, \ldots, x_n$ are the n unknowns and we have n equations. Usually a square matrix A means only one solution to $Ax = b$ (but not always). We can find x by geometry or by algebra.

This section begins with the row and column pictures of $Ax = b$. Then we solve the equations by simplifying them—eliminate x_1 from $n - 1$ equations to get a smaller system $A_2x_2 = b_2$ of size $n - 1$. Eventually we reach the 1 by 1 system $A_nx_n = b_n$ and we know $x_n = b_n/A_n$. Working backwards produces x_{n-1} and eventually we know x_2 and x_1.

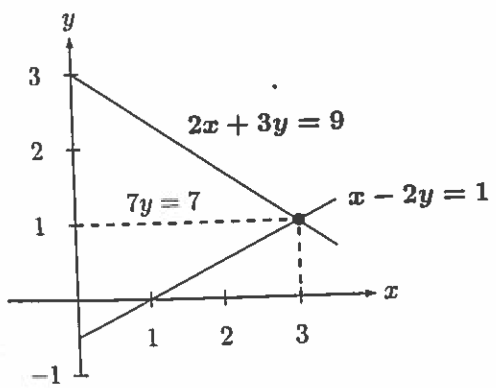
The point of this section is to see those elimination steps in terms of rank 1 matrices. **Every step (from A to A_2 and eventually to A_n) removes a matrix ℓu^*.** Then the original A is the sum of those rank one matrices. This sum is exactly the great factorization $A = LU$ into lower and upper triangular matrices L and U—as we will see.

$A = L$ times U is the matrix description of elimination without row exchanges. That will be the algebra. Start with geometry for this 2 by 2 example.

2 equations and 2 unknowns
2 by 2 matrix in $Ax = b$

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \qquad \begin{matrix} x - 2y = 1 \\ 2x + 3y = 9 \end{matrix} \qquad (1)$$

Notice! I multiplied Ax using inner products (dot products). Each row of the matrix A multiplied the vector x. That produced the two equations for x and y, and the two straight lines in Figure I.4. They meet at the solution $x = 3, y = 1$. Here is the **row picture**.



$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

$$A \quad x \quad = \quad b$$

At the solution

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

Figure I.4: **The row picture of $Ax = b$**: Two lines meet at the solution $x = 3, y = 1$.

Figure I.4 also includes the horizontal line $7y = 7$. I subtracted 2 (equation 1) from (equation 2). The unknown x has been eliminated from $7y = 7$. This is the algebra:

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 1 & -2 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \qquad \begin{matrix} x = 3 \\ y = 1 \end{matrix}$$

Again that elimination step removed a rank one matrix $\ell_1 u_1^*$. But A_2 is in a new place.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 7 \\ 2 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 8 \end{bmatrix} + \begin{bmatrix} 0 & \boxed{\begin{matrix} 1 & 1 \\ 1 & 3 \end{matrix}} \\ 0 & \\ 0 & 0 \quad 0 \end{bmatrix} \leftarrow A_2 \tag{10}$$

Elimination on A_2 produces two more rank one pieces. Then $A = LU$ has three pieces:

$$\ell_1 u_1^* + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}. \tag{11}$$

That last matrix U is triangular but the L matrix is not! The pivot order for this A was 3,1,2. If we want the pivot rows to be 1, 2, 3 we must move row 3 of A to the top:

Row exchange by a permutation P

$$PA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 7 \\ 2 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 1 & 1 \\ 1 & 3 & 7 \end{bmatrix}$$

When both sides of $Ax = b$ are multiplied by P, **order is restored and $PA = LU$:**

$$PA = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 1 & 1 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = LU. \tag{12}$$

Every invertible n by n matrix A leads to $PA = LU$: P = permutation.

There are six 3 by 3 permutations: Six ways to order the rows of the identity matrix.

1 exchange (odd P)

$$P_{213} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{321} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_{132} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

0 or 2 exchanges (even P)

$$P_{123} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad P_{312} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad P_{231} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The inverse of every permutation matrix P is its transpose P^{T}. The row exchanges will also apply to the right hand side b if we are solving $Ax = b$. The computer just remembers the exchanges without actually moving the rows.

There are $n!$ (n factorial) permutation matrices of size n: $3! = (3)(2)(1) = 6$. When A has dependent rows (no inverse), elimination leads to a zero row and stops short.

Problem Set I.4

1 Factor these matrices into $A = LU$:

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

2 If $a_{11}, \ldots, a_{1n}$ is the first row of a rank-1 matrix A and $a_{11}, \ldots, a_{m1}$ is the first column, *find a formula for a_{ij}*. Good to check when $a_{11} = 2, a_{12} = 3, a_{21} = 4$. When will your formula break down?

3 What lower triangular matrix E puts A into upper triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

4 **This problem shows how the one-step inverses multiply to give L.** You see this best when $A = L$ is already lower triangular with 1's on the diagonal. Then $U = I$:

$$\text{Multiply } A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \text{ by } E_1 = \begin{bmatrix} 1 & & \\ -a & 1 & \\ -b & 0 & 1 \end{bmatrix} \text{ and then } E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}.$$

(a) Multiply $E_2 E_1$ to find the single matrix E that produces $EA = I$.

(b) Multiply $E_1^{-1} E_2^{-1}$ to find the matrix $A = L$.

The multipliers a, b, c are mixed up in $E = L^{-1}$ but they are perfect in L.

5 *When zero appears in a pivot position, $A = LU$ is not possible!* (We are requiring nonzero pivots in U.) Show directly why these LU equations are both impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \ell & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ & f & h \\ & & i \end{bmatrix}.$$

These matrices need a row exchange by a *permutation matrix P*.

6 Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ will not be possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

7 (*Recommended*) Compute L and U for this symmetric matrix A:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four nonzero pivots.

8 *Tridiagonal matrices* have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into $A = LU$. Symmetry further produces $A = LDL^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

9 *Easy but important.* If A has pivots $5, 9, 3$ with no row exchanges, what are the pivots for the upper left 2 by 2 submatrix A_2 (without row 3 and column 3)?

10 Which invertible matrices allow $A = LU$ (elimination without row exchanges)? *Good question*! Look at each of the square upper left submatrices $A_1, A_2, \ldots, A_n$.

All upper left submatrices A_k must be invertible: sizes 1 by 1, 2 by 2, ..., n by n.

Explain that answer: A_k factors into ____ because $LU = \begin{bmatrix} L_k & 0 \\ * & * \end{bmatrix} \begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}$.

11 In some data science applications, the first pivot is the *largest number* $|a_{ij}|$ *in* A. Then row i becomes the first pivot row u_1^*. Column j is the first pivot column. Divide that column by a_{ij} so ℓ_1 has 1 in row i. Then remove that $\ell_1 u_1^*$ from A.

This example finds $a_{22} = 4$ as the first pivot ($i = j = 2$). Dividing by 4 gives ℓ_1:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} + \begin{bmatrix} -1/2 & 0 \\ 0 & 0 \end{bmatrix} = \ell_1 u_1^* + \ell_2 u_2^* = \begin{bmatrix} 1/2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1/2 & 0 \end{bmatrix}$$

For this A, both L and U involve permutations. P_1 exchanges the rows to give L. P_2 exchanges the columns to give an upper triangular U. Then $\boldsymbol{P_1 A P_2 = LU}$.

$$\textbf{Permuted in advance} \qquad P_1 A P_2 = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Question for $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$: Apply complete pivoting to produce $P_1 A P_2 = LU$.

12 If the short wide matrix A has $m < n$, how does elimination show that there are nonzero solutions to $Ax = 0$? What do we know about the dimension of that "nullspace of A" containing all solution vectors x? The nullspace dimension is at least ____.

Suggestion: First create a specific 2 by 3 matrix A and ask those questions about A.

I.5 Orthogonal Matrices and Subspaces

The word **orthogonal** appears everywhere in linear algebra. It means *perpendicular*. Its use extends far beyond the angle between two vectors. Here are important extensions of that key idea:

1. **Orthogonal vectors x and y.** The test is $x^{\mathrm{T}} y = x_1 y_1 + \cdots + x_n y_n = 0$.
   If x and y have complex components, change to $\overline{x}^{\mathrm{T}} y = \overline{x}_1 y_1 + \cdots + \overline{x}_n y_n = 0$.

2. **Orthogonal basis for a subspace:** Every pair of basis vectors has $v_i^{\mathrm{T}} v_j = 0$.
   Orthonormal basis: Orthogonal basis of **unit vectors**: every $v_i^{\mathrm{T}} v_i = 1$ (length 1).
   From orthogonal to orthonormal, just divide every basis vector v_i by its length $||v_i||$.

3. **Orthogonal subspaces R and N.** Every vector in the space **R** is orthogonal to every vector in **N**. Notice again! The row space and nullspace are orthogonal:

$$\begin{array}{c} Ax = 0 \text{ means} \\ \text{each row} \cdot x = 0 \end{array} \qquad \begin{bmatrix} \text{row 1 of } A \\ : \\ \text{row } m \text{ of } A \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ : \\ 0 \end{bmatrix}. \qquad (1)$$

   Every row (and every combination of rows) is orthogonal to all x in the nullspace.

4. **Tall thin matrices Q with orthonormal columns: $Q^{\mathrm{T}} Q = I$.**

$$Q^{\mathrm{T}} Q = \begin{bmatrix} — & q_1^{\mathrm{T}} & — \\ & : & \\ — & q_n^{\mathrm{T}} & — \end{bmatrix} \begin{bmatrix} \\ q_1 \cdots q_n \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \qquad (2)$$

   If this Q multiplies any vector x, the length of the vector does not change:

$$||Qx|| = ||x|| \quad \text{because} \quad (Qx)^{\mathrm{T}}(Qx) = x^{\mathrm{T}} Q^{\mathrm{T}} Q x = x^{\mathrm{T}} x \qquad (3)$$

   If $m > n$ the m rows cannot be orthogonal in $\mathbf{R}^n$. Tall thin matrices have $QQ^{\mathrm{T}} \neq I$.

5. **"Orthogonal matrices" are square with orthonormal columns: $Q^{\mathrm{T}} = Q^{-1}$.**
   For square matrices $Q^{\mathrm{T}} Q = I$ leads to $QQ^{\mathrm{T}} = I$
   For square matrices Q, the left inverse Q^{T} is also a right inverse of Q.
   The columns of this orthogonal n by n matrix are an orthonormal basis for $\mathbf{R}^n$.
   The rows of Q are a (probably different) orthonormal basis for $\mathbf{R}^n$.
   The name "orthogonal matrix" should really be "orthonormal matrix".

The next pages give examples of orthogonal **vectors, bases, subspaces** and **matrices**.

Orthogonal Basis = Orthogonal Axes in $\mathbf{R}^n$

Suppose the n by n orthogonal matrix Q has columns $q_1, \ldots, q_n$. Those unit vectors are a basis for n-dimensional space $\mathbf{R}^n$. Every vector v can be written as a combination of the basis vectors (the q's):

$$v = c_1 q_1 + \cdots + c_n q_n \qquad (13)$$

Those $c_1 q_1$ and $c_2 q_2$ and $c_n q_n$ are the components of v along the axes. They are the projections of v onto the axes! There is a simple formula for each number c_1 to c_n:

Coefficients in an orthonormal basis

$$\boxed{c_1 = q_1^{\mathrm{T}} v \qquad c_2 = q_2^{\mathrm{T}} v \quad \cdots \quad c_n = q_n^{\mathrm{T}} v} \qquad (14)$$

I will give a vector proof and a matrix proof. Take dot products with q_1 in equation (13):

$$q_1^{\mathrm{T}} v = c_1 q_1^{\mathrm{T}} q_1 + \cdots + c_n q_1^{\mathrm{T}} q_n = c_1 \qquad (15)$$

All terms are zero except $c_1 q_1^{\mathrm{T}} q_1 = c_1$. So $q_1^{\mathrm{T}} v = c_1$ and every $q_k^{\mathrm{T}} v = c_k$.

If we write (13) as a matrix equation $v = Qc$, **multiply by Q^{T} to see (14):**

$$Q^{\mathrm{T}} v = Q^{\mathrm{T}} Q c = c \quad \text{gives all the coefficients } c_k = q_k^{\mathrm{T}} v \text{ at once.}$$

This is the key application of orthogonal bases (for example the basis for Fourier series). When basis vectors are orthonormal, each coefficient c_1 to c_n can be found separately!

Householder Reflections

Here are neat examples of **reflection matrices** $Q = H_n$. Start with the identity matrix. Choose a unit vector u. Subtract the rank one symmetric matrix $2uu^{\mathrm{T}}$. Then $I - 2uu^{\mathrm{T}}$ is a "Householder matrix". For example, choose $u = (1, 1, \ldots, 1)/\sqrt{n}$.

Householder example

$$\boxed{H_n = I - 2uu^{\mathrm{T}} = I - \frac{2}{n}\text{ ones }(n, n).} \qquad (16)$$

With uu^{T}, H_n is surely symmetric. Two reflections give $H^2 = I$ because $u^{\mathrm{T}} u = 1$:

$$H^{\mathrm{T}} H = H^2 = (I - 2uu^{\mathrm{T}})(I - 2uu^{\mathrm{T}}) = I - 4uu^{\mathrm{T}} + 4uu^{\mathrm{T}} uu^{\mathrm{T}} = I. \qquad (17)$$

The 3 by 3 and 4 by 4 examples are easy to remember, and H_4 is like a Hadamard matrix:

$$H_3 = I - \frac{2}{3}\text{ ones} = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \qquad H_4 = I - \frac{2}{4}\text{ ones} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Householder's n by n reflection matrix has $H_n u = (I - 2uu^{\mathrm{T}})u = u - 2u = -u$. And $H_n w = +w$ whenever w is perpendicular to u. The "eigenvalues" of H are -1 (once) and $+1$ ($n-1$ times). All reflection matrices have eigenvalues -1 and 1.

Problem Set I.5

1 If u and v are orthogonal unit vectors, show that $u + v$ is orthogonal to $u - v$. What are the lengths of those vectors?

2 Draw unit vectors u and v that are *not* orthogonal. Show that $w = v - u(u^{\mathrm{T}} v)$ is orthogonal to u (and add w to your picture).

3 Draw any two vectors u and v out from the origin $(0, 0)$. Complete two more sides to make a parallelogram with diagonals $w = u + v$ and $z = u - v$. Show that $w^{\mathrm{T}} w + z^{\mathrm{T}} z$ is equal to $2u^{\mathrm{T}} u + 2v^{\mathrm{T}} v$.

4 Key property of every orthogonal matrix: $||Qx||^2 = ||x||^2$ for every vector x. More than this, show that $(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}} y$ for every vector x and y. So *lengths and angles are not changed by* Q. **Computations with Q never overflow!**

5 If Q is orthogonal, how do you know that Q is invertible and Q^{-1} is also orthogonal? If $Q_1^{\mathrm{T}} = Q_1^{-1}$ and $Q_2^{\mathrm{T}} = Q_2^{-1}$, show that $Q_1 Q_2$ is also an orthogonal matrix.

6 A **permutation matrix** has the same columns as the identity matrix (in some order). *Explain why this permutation matrix and every permutation matrix is orthogonal*:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ has orthonormal columns so } P^{\mathrm{T}} P = \underline{\qquad} \text{ and } P^{-1} = \underline{\qquad}.$$

When a matrix is symmetric or orthogonal, **it will have orthogonal eigenvectors.** This is the most important source of orthogonal vectors in applied mathematics.

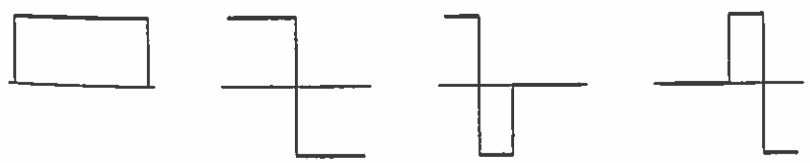
7 Four eigenvectors of that matrix P are $x_1 = (1, 1, 1, 1)$, $x_2 = (1, i, i^2, i^3)$, $x_3 = (1, i^2, i^4, i^6)$, and $x_4 = (1, i^3, i^6, i^9)$. Multiply P times each vector to find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The eigenvectors are the columns of the 4 by 4 **Fourier matrix** F.

$$\text{Show that } Q = \frac{F}{2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & i^2 & 1 & -1 \\ 1 & i^3 & -1 & i \end{bmatrix} \text{ has orthonormal columns : } \overline{Q}^{\mathrm{T}} Q = I$$

8 **Haar wavelets are orthogonal vectors (columns of W) using only 1, −1, and 0.**

$$n = 4 \qquad W = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

Find $W^{\mathrm{T}} W$ and W^{-1} and the eight Haar wavelets for $n = 8$.



Nondiagonalizable Matrices (Optional)

Suppose λ is an eigenvalue of A. We discover that fact in two ways:

1. **Eigenvectors** (geometric) There are nonzero solutions to $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

2. **Eigenvalues** (algebraic) The determinant of $A - \lambda I$ is zero.

The number λ may be a simple eigenvalue or a multiple eigenvalue, and we want to know its ***multiplicity***. Most eigenvalues have multiplicity $M = 1$ (simple eigenvalues). Then there is a single line of eigenvectors, and $\det(A - \lambda I)$ does not have a double factor.

For exceptional matrices, an eigenvalue can be ***repeated***. Then there are two different ways to count its multiplicity. Always GM $\leq$ AM for each λ:

1. **(Geometric Multiplicity = GM)** Count the **independent eigenvectors** for λ. Look at the dimension of the nullspace of $A - \lambda I$.

2. **(Algebraic Multiplicity = AM)** Count the **repetitions of** λ among the eigenvalues. Look at the roots of $\det(A - \lambda I) = 0$.

If A has $\lambda = 4, 4, 4$, then that eigenvalue has AM = 3 and GM = 1 or 2 or 3.

The following matrix A is the standard example of trouble. Its eigenvalue $\lambda = 0$ is repeated. It is a double eigenvalue (AM = 2) with only one eigenvector (GM = 1).

$$\begin{matrix}\textbf{AM} = \textbf{2}\\ \textbf{GM} = \textbf{1}\end{matrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2. \quad \begin{matrix}\boldsymbol{\lambda = 0, 0} \textbf{ but}\\ \textbf{1 eigenvector}\end{matrix}$$

There "should" be two eigenvectors, because $\lambda^2 = 0$ has a double root. The double factor λ^2 makes AM = 2. But there is only one eigenvector $\boldsymbol{x} = (1, 0)$. So GM = 1. ***This shortage of eigenvectors when* GM < AM *means that* A *is not diagonalizable.*** There is no invertible eigenvector matrix. The formula $A = X\Lambda X^{-1}$ fails.

These three matrices all have the same shortage of eigenvectors. Their repeated eigenvalue is $\lambda = 5$. Traces are 10 and determinants are 25:

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}.$$

Those all have $\det(A - \lambda I) = (\lambda - 5)^2$. The algebraic multiplicity is AM = 2. But each $A - 5I$ has rank $r = 1$. The geometric multiplicity is GM = 1. There is only one line of eigenvectors for $\lambda = 5$, and these matrices are not diagonalizable.

Problem Set I.6

1 The rotation $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has complex eigenvalues $\lambda = \cos\theta \pm i \sin\theta$:

$$Q\begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos\theta + i\sin\theta)\begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ and } Q\begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos\theta - i\sin\theta)\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Check that $\lambda_1 + \lambda_2$ equals the trace of Q (sum $Q_{11} + Q_{22}$ down the diagonal). Check that $(\lambda_1)(\lambda_2)$ equals the determinant. Check that those complex eigenvectors are orthogonal, using the complex dot product $\overline{\boldsymbol{x}}_1 \cdot \boldsymbol{x}_2$ (not just $\boldsymbol{x}_1 \cdot \boldsymbol{x}_2$!). What is Q^{-1} and what are its eigenvalues ?

2 Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the ____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues ____.

3 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A + B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

4 Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

5 (a) If you know that $\boldsymbol{x}$ is an eigenvector, the way to find λ is to ____.

(b) If you know that λ is an eigenvalue, the way to find $\boldsymbol{x}$ is to ____.

6 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain from those answers why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

7 **The determinant of A equals the product $\lambda_1\lambda_2\cdots\lambda_n$.** Start with the polynomial $\det(A-\lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda) \quad \text{so} \quad \det A = \_\_\_\_.$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

8 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A-\lambda I) = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a+d+\sqrt{})/2$ and $\lambda =$ ____. Their sum is ____. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A-\lambda I) =$ ____.

9 If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A-\lambda I) = (\lambda-4)(\lambda-5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20 and $\lambda = 4, 5$.

10 Choose the last rows of A and C to give eigenvalues 4, 7 and 1, 2, 3:

Companion matrices $$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

11 ***The eigenvalues of A equal the eigenvalues of A^{T}***. This is because $\det(A-\lambda I)$ equals $\det(A^{\mathrm{T}}-\lambda I)$. That is true because ____. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

12 This matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

13 Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Then $A = B$. *Reason*: Any vector $\boldsymbol{x}$ is a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. What is $A\boldsymbol{x}$? What is $B\boldsymbol{x}$?

14 Suppose A has eigenvalues 0, 3, 5 with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. If it did then ____ would be in the column space.

15 (a) Factor these two matrices into $A = X\Lambda X^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

(b) If $A = X\Lambda X^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.

16 Suppose $A = X\Lambda X^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\quad)(\quad)(\quad)^{-1}$.

17 True or false: If the columns of X (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) X is invertible (d) X is diagonalizable.

18 Write down the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

19 True or false: If the eigenvalues of A are 2, 2, 5 then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

20 True or false: If the only eigenvectors of A are multiples of (1, 4) then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $X\Lambda X^{-1}$.

21 $A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

22 Diagonalize A and compute $X\Lambda^k X^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{has} \quad A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

23 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = X\sqrt{\Lambda}X^{-1}$. Why is there no real matrix square root of B?

24 Suppose the same X diagonalizes both A and B. They have the *same eigenvectors* in $A = X\Lambda_1X^{-1}$ and $B = X\Lambda_2X^{-1}$. Prove that $AB = BA$.

25 The transpose of $A = X\Lambda X^{-1}$ is $A^{\mathrm{T}} = (X^{-1})^{\mathrm{T}}\Lambda X^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$ are the columns of that matrix $(X^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors of*** A, because $\boldsymbol{y}^{\mathrm{T}}A = \lambda\boldsymbol{y}^{\mathrm{T}}$. How do you multiply matrices to find this formula for A?

Sum of rank-1 matrices $A = X\Lambda X^{-1} = \lambda_1\boldsymbol{x}_1\boldsymbol{y}_1^{\mathrm{T}} + \cdots + \lambda_n\boldsymbol{x}_n\boldsymbol{y}_n^{\mathrm{T}}$.

26 **When is a matrix A similar to its eigenvalue matrix Λ?**

A and Λ always have the same eigenvalues. But similarity requires a matrix B with $A = B\Lambda B^{-1}$. Then B is the ____ matrix and A must have n independent ____.

Problem Set I.7

1 Suppose $S^{\mathrm{T}} = S$ and $Sx = \lambda x$ and $Sy = \alpha y$ are all real. Show that

$$y^{\mathrm{T}}Sx = \lambda y^{\mathrm{T}}x \quad \text{and} \quad x^{\mathrm{T}}Sy = \alpha x^{\mathrm{T}}y \quad \text{and} \quad y^{\mathrm{T}}Sx = x^{\mathrm{T}}Sy.$$

Show that $y^{\mathrm{T}}x$ must be zero if $\lambda \neq \alpha$: **orthogonal eigenvectors.**

2 Which of S_1, S_2, S_3, S_4 has two positive eigenvalues? Use a test, don't compute the λ's. Also find an x so that $x^{\mathrm{T}}S_1x < 0$, so S_1 is not positive definite.

$$S_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad S_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad S_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

3 For which numbers b and c are these matrices positive definite?

$$S = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \qquad S = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \qquad S = \begin{bmatrix} c & b \\ b & c \end{bmatrix}.$$

With the pivots in D and multiplier in L, factor each A into LDL^{T}.

4 Here is a quick "proof" that the eigenvalues of every real matrix A are real:

$$\textbf{False proof} \quad Ax = \lambda x \quad \text{gives} \quad x^{\mathrm{T}}Ax = \lambda x^{\mathrm{T}}x \quad \text{so} \quad \lambda = \frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} = \frac{\text{real}}{\text{real}}.$$

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the 90° rotation matrix $[\,0 \;\; -1; \;\; 1 \;\; 0\,]$ with $\lambda = i$ and $x = (i, 1)$.

5 Write S and B in the form $\lambda_1 x_1 x_1^{\mathrm{T}} + \lambda_2 x_2 x_2^{\mathrm{T}}$ of the spectral theorem $Q\Lambda Q^{\mathrm{T}}$:

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \qquad (\text{keep } \|x_1\| = \|x_2\| = 1).$$

6 (Recommended) This matrix M is antisymmetric and also _____. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($\|Mx\| = \|x\|$ for every x so $\|\lambda x\| = \|x\|$ for eigenvectors.) Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{can only have eigenvalues } i \text{ or } -i.$$

7 Show that this A (**symmetric but complex**) has only one line of eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \quad \text{is not even diagonalizable: eigenvalues } \lambda = 0 \text{ and } 0.$$

$A^{\mathrm{T}} = A$ is not such a special property for complex matrices. The good property is $\overline{A}^{\mathrm{T}} = A$. Then all eigenvalues are real and A has n orthogonal eigenvectors.

8 This A is nearly symmetric. But its eigenvectors are far from orthogonal:

$$A = \begin{bmatrix} 1 & 10^{-15} \\ 0 & 1 + 10^{-15} \end{bmatrix} \quad \text{has eigenvectors} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [?]$$

What is the angle between the eigenvectors?

9 *Which symmetric matrices S are also orthogonal?* Then $S^{\mathrm{T}} = S$ and $S^{\mathrm{T}} = S^{-1}$.

(a) Show how symmetry and orthogonality lead to $S^2 = I$.

(b) What are the possible eigenvalues of S? Describe all possible Λ.

Then $S = Q\Lambda Q^{\mathrm{T}}$ for one of those eigenvalue matrices Λ and an orthogonal Q.

10 If S is symmetric, show that $A^{\mathrm{T}}SA$ is also symmetric (take the transpose of $A^{\mathrm{T}}SA$). Here A is m by n and S is m by m. Are eigenvalues of S = eigenvalues of $A^{\mathrm{T}}SA$?

In case A is square and invertible, $A^{\mathrm{T}}SA$ is called *congruent* to S. They have the same number of positive, negative, and zero eigenvalues: ***Law of Inertia***.

11 Here is a way to show that a is *in between* the eigenvalues λ_1 and λ_2 of S:

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad \begin{array}{l} \det(S - \lambda I) = \lambda^2 - a\lambda - c\lambda + ac - b^2 \\ \text{is a parabola opening upwards (because of } \lambda^2) \end{array}$$

Show that $\det(S - \lambda I)$ is negative at $\lambda = a$. So the parabola crosses the axis left and right of $\lambda = a$. It crosses at the two eigenvalues of S so they must enclose a.

The $n-1$ eigenvalues of A always fall between the n eigenvalues of $S = \begin{bmatrix} A & b \\ b^{\mathrm{T}} & c \end{bmatrix}$. Section III.2 will explain this interlacing of eigenvalues.

12 The energy $x^{\mathrm{T}}Sx = 2x_1x_2$ certainly has a saddle point and not a minimum at $(0, 0)$. What symmetric matrix S produces this energy? What are its eigenvalues?

13 Test to see if $A^{\mathrm{T}}A$ is positive definite in each case: A needs independent columns.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

14 Find the 3 by 3 matrix S and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

15 Compute the three upper left determinants of S to establish positive definiteness. Verify that their ratios give the second and third pivots.

$$\textbf{Pivots = ratios of determinants} \qquad S=\begin{bmatrix}2&2&0\\2&5&3\\0&3&8\end{bmatrix}.$$

16 For what numbers c and d are S and T positive definite? Test their 3 determinants:

$$S=\begin{bmatrix}c&1&1\\1&c&1\\1&1&c\end{bmatrix} \quad \text{and} \quad T=\begin{bmatrix}1&2&3\\2&d&4\\3&4&5\end{bmatrix}.$$

17 Find a matrix with $a>0$ and $c>0$ and $a+c>2b$ that has a negative eigenvalue.

18 A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have $x^{\mathrm{T}}Sx>0$:

$$\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}\begin{bmatrix}4&1&1\\1&0&2\\1&2&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \text{ is not positive when } (x_1,x_2,x_3)=(\quad,\quad,\quad).$$

19 A diagonal entry s_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $S-s_{jj}I$ would have _____ eigenvalues and would be positive definite. But $S-s_{jj}I$ has a _____ on the main diagonal, impossible by Problem 18.

20 From $S=Q\Lambda Q^{\mathrm{T}}$ compute the positive definite symmetric square root $Q\sqrt{\Lambda}Q^{\mathrm{T}}$ of each matrix. Check that this square root gives $A^{\mathrm{T}}A=S$:

$$S=\begin{bmatrix}5&4\\4&5\end{bmatrix} \quad \text{and} \quad S=\begin{bmatrix}10&6\\6&10\end{bmatrix}.$$

21 Draw the tilted ellipse $x^2+xy+y^2=1$ and find the half-lengths of its axes from the eigenvalues of the corresponding matrix S.

22 In the Cholesky factorization $S=A^{\mathrm{T}}A$, with $A=\sqrt{D}L^{\mathrm{T}}$, the square roots of the pivots are on the diagonal of A. Find A (upper triangular) for

$$S=\begin{bmatrix}9&0&0\\0&1&2\\0&2&8\end{bmatrix} \quad \text{and} \quad S=\begin{bmatrix}1&1&1\\1&2&2\\1&2&7\end{bmatrix}.$$

23 Suppose C is positive definite (so $y^{\mathrm{T}}Cy>0$ whenever $y\neq 0$) and A has independent columns (so $Ax\neq 0$ whenever $x\neq 0$). Apply the energy test to $x^{\mathrm{T}}A^{\mathrm{T}}CAx$ to show that $S=A^{\mathrm{T}}CA$ *is positive definite: the crucial matrix in engineering.*

The Minimum of a Function $F(x,y,z)$

What tests would you expect for a minimum point? First come zero slopes:

First derivatives are zero $\dfrac{\partial F}{\partial x}=\dfrac{\partial F}{\partial y}=\dfrac{\partial F}{\partial z}=0$ at the minimum point.

Next comes the linear algebra version of the usual calculus test $d^2f/dx^2>0$:

$$\textbf{Second derivative matrix } H \textbf{ is positive definite} \quad H=\begin{bmatrix}F_{xx}&F_{xy}&F_{xz}\\F_{yx}&F_{yy}&F_{yz}\\F_{zx}&F_{zy}&F_{zz}\end{bmatrix}$$

Here $F_{xy}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial F}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial F}{\partial x}\right)=F_{yx}$ is a 'mixed" second derivative.

24 For $F_1(x,y)=\frac{1}{4}x^4+x^2y+y^2$ and $F_2(x,y)=x^3+xy-x$ find the second derivative matrices H_1 and H_2 (the **Hessian matrices**):

$$\textbf{Test for minimum} \quad H=\begin{bmatrix}\partial^2F/\partial x^2 & \partial^2F/\partial x\partial y\\ \partial^2F/\partial y\partial x & \partial^2F/\partial y^2\end{bmatrix} \text{ is positive definite}$$

H_1 is positive definite so F_1 is concave up (= convex). Find the minimum point of F_1. *Find the saddle point of* F_2 (look only where first derivatives are zero).

25 Which values of c give a bowl and which c give a saddle point for the graph of $z=4x^2+12xy+cy^2$? Describe this graph at the borderline value of c.

26 Without multiplying $S=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}2&0\\0&5\end{bmatrix}\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$, find

(a) the determinant of S (b) the eigenvalues of S
(c) the eigenvectors of S (d) a reason why S is symmetric positive definite.

27 For which a and c is this matrix positive definite? For which a and c is it positive semidefinite (this includes definite)?

$$S=\begin{bmatrix}a&a&a\\a&a+c&a-c\\a&a-c&a+c\end{bmatrix}$$

All 5 tests are possible.
The energy $x^{\mathrm{T}}Sx$ equals
$a(x_1+x_2+x_3)^2+c(x_2-x_3)^2$.

28 **Important!** Suppose S is positive definite with eigenvalues $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$.

(a) What are the eigenvalues of the matrix $\lambda_1 I-S$? Is it positive semidefinite?

(b) How does it follow that $\lambda_1 x^{\mathrm{T}}x\geq x^{\mathrm{T}}Sx$ for every x?

(c) Draw this conclusion: **The maximum value of $x^{\mathrm{T}}Sx/x^{\mathrm{T}}x$ is λ_1.**

Note Another way to 28 (c): **Maximize $x^{\mathrm{T}}Sx$ subject to the condition $x^{\mathrm{T}}x=1$.**

This leads to $\dfrac{\partial}{\partial x}\left[x^{\mathrm{T}}Sx-\lambda\left(x^{\mathrm{T}}x-1\right)\right]=0$ and then $Sx=\lambda x$ and $\lambda=\lambda_1$.