Math 309 Intermediate Linear Algebra Exam 1 Sample solution

1. Give an example with explanation/verification for each of the following.

(a) A real matrix A with no real eigenvalues.

Example. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $\det(\lambda I - A) = \lambda^2 + 1$. So, the eigenvalues are $\pm i$.

(b) A square matrix $B \in M_n(\mathbb{C})$ has two different eigenvalues and is not diagonalizable.

Example. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Clearly, the eigenvalues are $0, 0, 1$. However, $Ax = 0x$ has only one linearly independent solution x, and $Ax = x$ has one linearly independent solution x. So, we cannot find 3 linearly independent eigenvectors for A. Thus, A is not diagonalizable.

(c) A 2×2 matrix C such that $C \neq XY$ for any lower triangular X and upper triangular Y.

Example. Let $C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. If $C = XY$ for some lower triangular $X = \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix}$ and $Y = \begin{pmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{pmatrix}$, then $x_{11} = 0$ or $y_{11} = 0$ so that XY has zero first row, or zero first column, which is impossible.

(d) A non-symmetric real matrix X such that the row space and column space are the same, and $N(X) = N(X^T)$.

Example. Let $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then both the row space and column space of X equal $\mathbb{R}^2$ and $N(X) = N(X^T) = \{0\}$.

2. Let $A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$.

(a) Determine the value c leading to zero in the second pivot positions.

The leading 1×1 minor is $1 \neq 0$. The leading 2×2 minor is $4 - 2c$, which is zero if and only if $c = 2$.

(b) For the value c found in (a), find a permutation matrix P so that $PA = LU$.

If $c = 2$, we can swap the second and third rows of A so that all the leading minors are nonzero. So, for

$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, LA will have an LU factorization.

(c) If $c = 1$, find the LU factorization of A.

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{pmatrix}$. Then $A - A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$ for $A_1 = u_1 v_1^t = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$.

Now, $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix}$. So, $A = LU$ with $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

3. Find the singular value decomposition of each of the following matrices, i.e., write the matrix as $\sum_{j=1}^{k} s_j u_j v_j^*$ for positive numbers $s_1, \ldots, s_k$, orthnormal families $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$.

   (a) $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

   Not that $A^T A = \begin{pmatrix} 5 & 3 & 3 & 9 \end{pmatrix}$. Solving $\det(\lambda I - A^t A) = \lambda^2 - 14\lambda + 36 = 0$, we see that $\lambda = 7 \pm \sqrt{3}$ are the eigenvalues. We get an eigenvector $x_1 = \begin{pmatrix} 2 - \sqrt{13} \\ 3 \end{pmatrix}$ for $\lambda_1 = 7 + \sqrt{13}$ and an eigenvector $x_2 = \begin{pmatrix} 2 + \sqrt{13} \\ 3 \end{pmatrix}$ for $\lambda_2 = 7 - \sqrt{13}$. Now, $y_1 = Ax_1 = \begin{pmatrix} 2 - \sqrt{13} \\ 11 + \sqrt{13} \end{pmatrix}$ and $y_2 = Ax_2 = \begin{pmatrix} 2 - \sqrt{13} \\ 9 - \sqrt{13} \end{pmatrix}$. Let $v_1 = \frac{x_1}{\|x_1\|}, v_2 = \frac{x_2}{\|x_2\|}, u_1 = \frac{y_1}{\|y_1\|}, u_2 = \frac{y_2}{\|y_2\|}$, $s_1 = \sqrt{\lambda_1}, s_2 = \sqrt{\lambda_2}$.

   Then $A[v_1 v_2] = [u_1 u_2]D$ with $D = \mathrm{diag}\,(s_1, s_2)$. Hence, $A = s_1 u_1 v_1^t + s_2 u_2 v_2^t$.

   (b) $B = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -2 & -5 \\ 2 & 4 & 10 \end{bmatrix}$.

   $B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} (1\ 2\ 5) = u_1 s_1 v_1$ with $u_1 = (1, -1, 2)^T/\sqrt{6}$, $v_1 = (1, 2, 5)^T/\sqrt{30}$, and $s_1 = 6\sqrt{5}$.

4. Recall that $A \in M_n(\mathbb{C})$ is positive definite if $A = A^*$ and A has positive eigenvalues, i.e., $A = UDU^*$ for a unitary matrix U and a diagonal matrix with positive diagonal entries.

   (a) Show that if $A \in M_n(\mathbb{C})$ is positive definite, then $A = B^2$ for a positive definite matrix B.

   If $A = A^*$ have positive eigenvalues, then $A = UDU^*$ for some unitary matrix U and diagonal matrix $D = \mathrm{diag}\,(d_1, \ldots, d_n)$ with positive diagonal entries. So, $A = B^2$ with $B = UD^{1/2}U^*$, where $D^{1/2} = \mathrm{diag}\,(\sqrt{d_1}, \ldots, \sqrt{d_n})$.

   (b) If $A = B^2$ for a positive definite matrix B, show that $x^*Ax > 0$ for any nonzero vector $x \in \mathbb{C}^n$.

   If $A = B^2 = B^*B$ where B is positive definite, then for any nonzero vector x,

   $$x^*Ax = x^*B^*Bx = \|Bx\|^2 > 0.$$

   (c) Show that if $A = A^*$ and $x^*Ax > 0$ for every nonzero vector $x \in \mathbb{C}^n$, then all eigenvalues of A are positive.

   Suppose λ is an eigenvalue of A with an eigenvector of unit length x.

   We have $\lambda = x^*(\lambda x) = x^*Ax > 0$. So, λ is positive.

5. (a) Suppose $A = S^{-1}BS$, where $A, B, S \in M_n$. Show that A and B have the same characteristic polynomials, hence the same eigenvalues.

$\det(\lambda I - A) = \det(\lambda S^{-1}S - S^{-1}BS) = \det(S^{-1}(\lambda I - B)S) = \det(S^{-1})\det(\lambda I - B)\det(S)$
$= \det(S)\det(S^{-1})\det(\lambda I - B) = \det(SS^{-1})\det(\lambda I - B) = \det(\lambda I - B)$.

So, A and B have the same characteristic polynomials, hence the same eigenvalues.

(b) Show that for any $X \in M_{m,n}, Y \in M_{n,m}$, $\begin{bmatrix} I_m & X \\ 0 & I_n \end{bmatrix}^{-1} \begin{bmatrix} XY & 0 \\ Y & 0_n \end{bmatrix} \begin{bmatrix} I_m & X \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} 0_m & 0 \\ Y & YX \end{bmatrix}$.

One can check that $\begin{bmatrix} XY & 0 \\ Y & 0_n \end{bmatrix} \begin{bmatrix} I_m & X \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_m & X \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0_m & 0 \\ Y & YX \end{bmatrix}$. The result follows.

(c) Use (a) and (b) to conclude that for any $X \in M_{m,n}, Y \in M_{n,m}$ the matrices XY and YX

Because $\begin{bmatrix} XY & 0 \\ Y & 0_n \end{bmatrix}$ and $\begin{bmatrix} 0_m & 0 \\ Y & YX \end{bmatrix}$ are similar, they have the same eienvalues. The nonzero eigenvalues of the left matrices are those of XY, and the nonero eigenvalues of the right matrices are those of YX. The result follows. have the same nonzero eigenvalues.