Math 309 Intermediate Linear Algebra Exam 2 Name: Sample Solution

Both handwritten and typed solution will be accepted.

1. Let

$$S = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

(a) Compute the generalized eigenvalues λ_1, λ_2 and generalized eigenvectors x_1, x_2 such that

$$(S - \lambda_j M)x_j \text{ for } j = 1, 2.$$

(b) Show that $x_1^T M x_2 = 0$.

(c) Change M to $M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Show that $(S - \lambda M)$ has an eigenvalue ∞, and the two generalized eigenvectors are still M-orthogonal to each other.

Solution. (a) We solve

$$0 = \det(S - \lambda M) = \det\left(\begin{bmatrix} 3-\lambda & -2 \\ -2 & 3-4\lambda \end{bmatrix}\right) = (3-\lambda)(3-4\lambda) - 4 = 4\lambda^2 - 15\lambda - 5,$$

and get $\lambda = \frac{1}{8}(15 \pm \sqrt{145})$.

If $\lambda_1 = \frac{1}{8}(15+\sqrt{145})$ and $\begin{bmatrix} 3 - \frac{1}{8}(15+\sqrt{145}) & -2 \\ -2 & 3 - \frac{1}{2}(15+\sqrt{145}) \end{bmatrix} x_1 = 0$, then $x_1 = \begin{bmatrix} 16 \\ 9 - \sqrt{145} \end{bmatrix}$;

If $\lambda_1 = \frac{1}{8}(15-\sqrt{145})$ and $\begin{bmatrix} 3 - \frac{1}{8}(15-\sqrt{145}) & -2 \\ -2 & 3 - \frac{1}{2}(15-\sqrt{145}) \end{bmatrix} x_2 = 0$, then $x_2 = \begin{bmatrix} 16 \\ 9 + \sqrt{145} \end{bmatrix}$.

(b) $X_1^T M X_2 = \begin{bmatrix} 16 & 9 - \sqrt{145} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 16 \\ 9 + \sqrt{145} \end{bmatrix} = 16^2 + 4(81 - 145) = 4(64 + 81 - 145) = 0.$

(c) If $M = \text{diag}\,(1, 0)$, then $0 = \det(S - \lambda M) = \det\left(\begin{bmatrix} 3-\lambda & -2 \\ -2 & 3 \end{bmatrix}\right) = 5 - 3\lambda$ so that $\lambda = 5/3$.

We solve $0 = (S - \frac{5}{3}M)x_1 = \begin{bmatrix} 3-\lambda & -2 \\ -s & 3 \end{bmatrix} x_1 \begin{bmatrix} 16 \\ 9 + \sqrt{145} \end{bmatrix}$, and get $x_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

For $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $Sx \neq 0$ and $Mx = 0$. Thus, $\alpha S x_2 = \beta M x$ with $(\alpha, \beta) = (0, 5/3)$.

Thus, $\lambda_2 = \beta/\alpha = \infty$ and $x_1^T M x_2 = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.$

Common mistakes

(a) Mistake in calculating x_1, x_2.

(b) Verify $x_1^T M x_2 = 0$. One does not just cite or reprove the result. One needs to verify it for this specific problem. Imagine you need to apply the result in your future work, you do not just cite the result. You have to verify your computational answer to make sure that it agrees with the theory!

(c) Eigenvector/generalized eigenvectors have to be nonzero. So, $x_2 \neq 0$.

2. Let $A = \begin{bmatrix} 3 & 0 \\ 4 & 5i \end{bmatrix}$.

(a) Determine $\|A\|_2$ and $x \in \mathbb{C}^2$ such that $\ell^2(x) = 1$ and $\|A\|_2 = \ell^2(Ax)$.

(b) Determine $x \in \mathbb{C}^2$ such that $\ell^1(x) = 1$ and $\|A\|_1 = \ell^1(Ax)$.

(c) Determine $x \in \mathbb{C}^2$ such that $\ell^\infty(x) = 1$ and $\|A\|_\infty = \ell^\infty(Ax)$.

(d) Give an example B such that $\|AB\|_2 \neq \|BA\|_2$.

Solution. (a) Solve

$0 = \det(\lambda I - A^*A) = \det\left(\begin{bmatrix} \lambda - 25 & -20i \\ +20i & \lambda - 25 \end{bmatrix}\right) = (\lambda - 25)^2 - 400 = (\lambda - 45)(\lambda - 5)$.

Moreover, if $0 = (45I - A^*A)x = \begin{bmatrix} 20 & -20i \\ +20i & 20 \end{bmatrix}$, then $x = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$.

We see that $\|A\|_2 = \sqrt{45}$ and $x = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$ satisfies $\ell^2(x) = 1$ and $\ell^2(Ax) = \|A\|_2$.

(b) Note: $\ell^1(A) = \max\{|3| + |4|, |5i|\} = 7$; $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ satisfies $\ell^1(x) = 1$ and $\ell^1(Ax) = \|A\|_1$.

(c) Note: $\ell^\infty(A) = \max\{|3|, |4| + |5i|\} = 9$; $x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ satisfies $\ell^\infty(x) = 1$ and $\ell^\infty(Ax) = \|A\|_\infty$.

(d) Let $B = \operatorname{diag}(1, 0)$. Then $BA = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$ and $AB = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$ so that $\|BA\|_2 = 3$ and $AB = 5$ as $(AB)^*(AB) = \operatorname{diag}(25, 0)$.

Common mistakes (a) Fail to explain how to determine $\|A\|_2 = \sqrt{45}$.

(b) and (c): Miscalculation of the norms and vectors.

3. Let $A = s_1 u_1 v_1^T + s_2 u_2 v_2^T$, where $(s_1, s_2) = (2, 1)$,

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Find the best rank one approximation A_1 of A with respect to the Frobenius norm.

(b) Find matrices U_1, V_1 with orthonormal columns such that $A = U_1 \Sigma_1 V_1^T$ with $\Sigma_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) Find matrices U_2, V_2 with orthonormal columns such that $A = U_2 \Sigma_2 V_2^T$ with $\Sigma_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Solution. (a) $A_1 = 2u_1 v_1^*$ is the best rank one approximation of A.

(b) Let $U_1 = [u_1 \ u_2]$ and $V_1 = [v_1 \ v_2]$. Then $A = U_1 \Sigma_1 V_1^T$.

(c) Let $U_2 = [u_1 \ u_2 \ u_3]$ and $V_1 = [v_1 \ v_2 \ v_3 \ v_4]$ with

$$u_3 = [1, 1, -2]^T / \sqrt{6}, \ v_3 = [1, 1, -2, 0]^T / \sqrt{6}, \ v_4 = [0, 0, 0, 1]^T.$$

Then $U_2^* U_2 = I_3$ and $V_2^* V_2 = I_4$ so that U_2, V_2 have orthonormal columns, and $A = U_2 \Sigma_2 V_2^T$.

Common mistake: Fail to construct orthonormal columns in U_2, V_2.

4. Let $A \in M_n$ be a complex matrix.

(a) Suppose λ_1 is an egenvalue of A with a unit eigenvector u_1, $U_1 \in M_n$ is unitary with u_1 as the first column. Show that $U_1^* A U_1$ has the form $\begin{bmatrix} \lambda_1 & x^* \\ 0 & A_1 \end{bmatrix}$, where $x \in \mathbb{C}^{n-1}$ and $A_1 \in M_{n-1}$.

[Hint: Argue that the first column of $U_1^* A U_1$ has the form $(\lambda_1, 0, \ldots, 0)^T$.]

(b) Prove by induction that there is a unitary $U \in M_n$ such that U^*AU is in upper triangular form, i.e., all the (i, j) entries are zero if $i > j$.

[Hint: If $n > 1$, apply induction assumption to the matrix A_2 in part (a) to get a unitary $U_2 \in M_2$ such that $U_2^* A_2 U_2$ is in upper triangular form. Let $U = U_1 \begin{bmatrix} 1 & \\ & U_2 \end{bmatrix}$. Then ...]

Solution. (a) Suppose $U_1 = [u_1 \cdots u_n]$. Then the first column of AU_1 is $\lambda_1 u_1$. Because $u_1^* \lambda_1 u_1 = \lambda_1$ and $u_j^* \lambda_1 u_1 = 0$ for $j = 2, \ldots, n$, we see that $U^* A u_1 = [\lambda_1, 0, \ldots, 0]^T$, which is the first column of U^*AU. Hence, U^*AU has the asserted form.

(b) We prove the statement by induction on n. If $n = 1$, then $A = [\lambda_1]$, and the result holds.

Suppose $n > 1$. By part (a), there is a unitary U_1 so that $U_1^* A U_1$ has the asserted form. By induction assumption, there is a unitary $U_2 \in M_{n-1}$ such that $U_2^* A_1 U_2$ is in upper triangular form T_1. If $U = U_1 \begin{bmatrix} 1 & \\ & U_2 \end{bmatrix}$. Then U^*AU equals

$$\begin{bmatrix} 1 & \\ & U_2^* \end{bmatrix} U_1^* A U_1 \begin{bmatrix} 1 & \\ & U_2 \end{bmatrix} = \begin{bmatrix} 1 & \\ & U_2^* \end{bmatrix} \begin{bmatrix} \lambda_1 & x^* \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} 1 & \\ & U_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & x^* U_2 \\ 0 & U_2^* A_1 U_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & x^* U_2 \\ 0 & T_1 \end{bmatrix}$$

is in upper triangular form. By the principle of M.I. the statement holds for all n.

Common mistakes: (a) Fail to explain clearly the form of the first column of $U_1^* A U_1$.

(b) Fail to explain clearly why U^*AU will be in upper triangular form.

5. Suppose $A = SDS^{-1}$ such that D is a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.

(a) Show that $\det(\lambda I - A) = \det(\lambda I - S^{-1}DS)$.

(b) For any polynomial $f(x) = x^m + f_{m-1}x^{m-1} + \cdots + f_0$, and $C \in M_n$, let

$$f(C) = C^m + f_{m-1}C^{m-1} + \cdots + f_0 I_n.$$

Prove that $f(A) = A^m + f_{m-1}A^{m-1} + \cdots + f_0 I_n = S(D^m + f_{m-1}D^{m-1} + \cdots + f_0 I_n)S^{-1}$.

[Hint: First show that $A^k = SD^kS^{-1}$ for any positive integer k.]

(c) Show that if $f(x) = \det(xI - A) = (x - \lambda_1)\cdots(x - \lambda_n)$, then

$$f(A) = (A - \lambda_1 I)\ldots(A - \lambda_n I) = 0.$$

[Hint: Show that $f(D) = (D - \lambda_1 I)\cdots(D - \lambda_n I) = 0$ and use (b).]

Solution. (a) We have

$$\begin{aligned}
\det(\lambda I - A) &= \det(\lambda SIS^{-1} - SDS^{-1}) = \det(S(\lambda I - D)S^{-1}) \\
&= \det(S)\det(\lambda I - D)\det(S^{-1}) \\
&= \det(\lambda I - D) \qquad\qquad \text{because } \det(S)\det(S^{-1} = 1.
\end{aligned}$$

(b) Note that for any positive integer k, $A^k = (SDS^{-1})^k = (SDS^{-1})\cdots(SDS^{-1}) = SD^kS^{-1}$. Thus,

$$\begin{aligned}
f(A) &= A^m + f_{m-1}A^{m-1} + \cdots + f_0 I \\
&= SD^mS^{-1} + f_{m-1}SD^{m-1}S^{-1} + \cdots + f_0 SIS^{-1} \\
&= S(D^m + f_{m-1}D^{m-1} + \cdots + f_0 I)S^{-1}.
\end{aligned}$$

(c) Note that $f(D) = (D - \lambda_1 I)\cdots(D - \lambda nI)$. and $D_j = D - \lambda_j I$ is a diagonal matrix with the jth diagonal entries equal to 0. Now, the kth diagonal entry of $f(D) = D_1 \cdots D_n$ is the product of the kth diagonal entries of those of $D_1, \ldots, D_n$. Hence, all diagonal entries of $f(D)$ are zero. It follows that $f(D) = 0$. By (b), $f(A) = Sf(D)S^{-1} = S0S^{-1} = 0$.

Common mistakes Explanation not clear.

Extra credit for making up Exam 1 score

Suppose $A = UTU^* \in M_n$ is a complex matrix such that U is unitary and $T = [t_{ij}]$ is an upper triangular matrix. Let m be a positive integer. Show that there are $d_1, \ldots, d_n \in \mathbb{C}$ such that

(a) $(t_{11} + d_1, \ldots, t_{nn} + d_n)$ has n distinct entries and $|d_1|^2 + \cdots + |d_n|^2 < 1/m^2$, and

[Hint: Let $d_1 = 0$; for $j > 1$ let d_j with $|d_j| < 1/(m^2 n)$, and $t_{jj} + d_j \notin \{t_{ii} + d_i : 1 \le i < j\}$.]

(b) if $D = \operatorname{diag}(d_1, \ldots, d_n)$ and $\tilde{A} = A + UDU^*$, then $\|\tilde{A} - A\|_F < 1/m$.

Solution. (a) Let $d_1 = 0$. For $j = 1$ we can choose d_2 with $|d_2| < 1/(m^2 n)$, and $t_{22} + d_2 \neq t_{11}$. For $2 \le j < n$, suppose $d_1, \ldots, d_{j-1}$ are chosen so that $t_{11} + d_1, \ldots, t_{jj} + d_j$ are distinct, where $d_\ell| \le 1/(m^2 n)$ for $\ell = 1, \ldots, j$. Since there are infinitely many numbers d with $|d| < 1/(m^2 n)$, we can find d_{j+1} such that $t_{j+1,j+1} + d_{j+1} \notin \{t_{\ell\ell} + d_\ell : 1 \le \ell \le j\}$. We can repeat this argument until we get $d_1, \ldots, d_n$ with the asserted property.

(b) Note that $\|\tilde{A} - A\|_F = UDU^*\|_F = \|D\|_F = (\sum_{j=1}^n |d_j|^2)^{1/2} < (1/m^2)^{1/2} = 1/m$.

Remark

By Problem 5, if A is diagonalizable and $f(\lambda) = \det(\lambda I - A)$, then $f(A)$ is the zero matrix.

In general, by Problem 4, $A = UTU^*$ for some upper triangular matrix. Now, for every positive integer m, there is $A_m = U(T + D_m)U^*$ for some diagonal matrix D_m so that $\|A_m - A\|_F < 1/m$. Now, $f_m(\lambda) = \det(\lambda I - A_m)$. Since A_m has n distinct eigenvalues and is diagonalizable, we have $f_m(A_m) = 0$. As $m \to \infty$, $f_m(\lambda) \to f(\lambda)$ and $A_m \to A$. Thus, $f(A) = 0$. So, we have proved the Cayley-Hamilton Theorem.

Theorem *If $A \in M_n$ and $f(\lambda) = \det(\lambda I - A)$, then $f(A)$ is the zero matrix in M_n.*

Also, we have show that if $A \in M_n$ is diagonalizable, and $\lambda_{\max}(A) < 1$, then $A^k \to 0$ as $k \to \infty$. Now, if $\lambda_{\max}(A) < 1$ and A is not diagonazable, then we can find $A_m \in M_n$ with distinct eigenvalues and $\lambda_{\max}(A_m) < \lambda_{\max}(A) + 1/m$. So, for sufficiently large m, we have $\lambda_{\max}(A_m) < 1$ so that $A_m^k \to 0$ as $k \to \infty$. Since $A_m \to A$, $A_m^k \to A^k$, and hence $A^k \to 0$. This complete the proof of the following.

Theorem A matrix $A \in M_n$ satisfies $A^k \to 0$ as $k \to \infty$ if and only if $\lambda_{\max}(A) < 1$.