Math 309 Intermediate Linear Algebra **Final Exam** Name: ______________________

Send your (typed or handwritten) solution to ckli@math.wm.edu by 11:59 p.m. May 13.

1. (a) Determine A^{-1} if $A = \begin{bmatrix} 1 & 0 & t \\ 0 & 2 & t^2 \\ 0 & 0 & 4 \end{bmatrix}$ for $t \in \mathbb{R}$, and determine $\frac{dA^{-1}}{dt}$ by differentiating the entries of A^{-1} AND also by the formula $-A^{-1}\frac{dA}{dt}A^{-1}$

(b) Let $\lambda_1(t), \lambda_2(t)$ be the eigenvalues of $B = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} + t \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ for $t \in \mathbb{R}$. Determine $\frac{d\lambda_j}{dt}$ at $t = 0$.

2. Find a degree 2 polynomial $f(x)$ such that $f(0) = 1,\ f(1) = 2,\ f(2) = 7$.

3. Let $A \in M_{m,n}$ with nonzero singular values $s_1 \ge \cdots \ge s_r > 0$. Show that

$$\begin{bmatrix} 0_m & A \\ A^* & 0_n \end{bmatrix}$$

have nonzero eigenvalues $\pm s_1, \dots, \pm s_r$.

4. Suppose $H = H^* \in M_n$ have eigenvalues $\lambda_1 \ge \dots, \ge \lambda_n$. Let $H_k \in M_k$ be a principal submatrix of H with eigenvalues $\mu_1 \ge \cdots \ge \mu_k$.

(a) Show that $\lambda_j \ge \mu_j$ for $j = 1, \dots, k$.

(b) Show that $\mathrm{tr}\,(H_k) \le \sum_{j=1}^k \lambda_j$.

(c) If H has diagonal entries $d_1 \ge \cdots \ge d_n$, Show that for $k = 1, \dots, n-1$,

$$d_1 + \cdots + d_k \le \lambda_1 + \cdots + \lambda_k.$$

5. Let $X, Y \in M_{m,n}$. Show that for $j = 1, \dots, n$,

$$\lambda_j(X^*X + Y^*Y) \ge \lambda_j(X^*Y + Y^*X).$$

[Hint: Consider $(X - Y)^*(X - Y)$.]

6. Recall that $A \in M_n$ is normal if and only if $AA^* = A^*A$. Suppose $T \in M_n$ is upper triangular. Show that T is normal if and only if T is a diagonal matrix.

7. Let $P = E_{12} + E_{23} + \cdots + E_{n-1,n} + E_{n1}$, $w = e^{i2\pi/n}$, and $V = \frac{1}{\sqrt{n}}[w^{(i-1)(j-1)}]$.

(a) Show that $V^*PV = D = \mathrm{diag}\,(1, w, w^2, \dots, w^{n-1})$.

(b) Let $A = a_0 I + a_1 P + \cdots + a_{n-1}P^{n-1}$. Show that $V^*AV = \sum_{j=0}^n a_j D^j$ and determine the eigenvalues of A.

8. Suppose $A \in M_{m,n}$ has rank r. Show that there are $C \in M_{m,r}, R \in M_{n,r}$ such that $A = CR$ and $\|A\|_N = \|C\|_F^2 = \|R\|_F^2$. [Hint: Let $A = U\Sigma V^*$. Then ...]

Extra credit problem for Exam 1 or 2.

9. Let $A \in M_n$.

(a) If A is singular, show that there is $j \in \{1, \ldots, n\}$ such that $|a_{jj}| \le \sum_{\ell \ne j} |a_{j\ell}|$.

[Hint: Suppose $x = [x_1, \ldots, x_n]^T \in \mathbb{F}^n$ is nonzero such that $Ax = 0$. If $|x_\ell| \le |x_j|$ for all ℓ, use the fact that $[a_{j1}, \ldots, a_{jn}]x = 0$ to deduce the conclusion.]

(b) Use (a) to deduce that if λ is an eigenvalue of A, then there is $i \in \{1, \ldots, n\}$ such that $|a_{ii} - \lambda| \le \sum_{\ell \ne i} |a_{i\ell}|$.

10. Let A be a partial matrix in $M_{m,n}$. Then a completion A_0 with minimum nuclear norm satisfies

$$\|A_0\|_N = \min\left\{[\mathrm{tr}\,(P_1) + \mathrm{tr}\,(P_2)]/2 : \begin{bmatrix} P_1 & A_0 \\ A_0^* & P_2 \end{bmatrix} \text{ is psd}\right\}.$$