MATH 309 HW10 Solution

III.1 #1 According to the problem, let $x = v^Tu$. Thus, we have $(I - uv^T)^{-1} = I + u(1 + x + x^2 + \cdots)v^T = I + u(1-x)^{-1}v^T = I + \frac{uv^T}{1-v^Tu}$.

III.1 #2 We compute E^{-1} in two ways: from (4), we have $E^{-1} = \begin{bmatrix} I & -\frac{u}{1-v^Tu} \\ 0 & \frac{1}{1-v^Tu} \end{bmatrix} \begin{bmatrix} I & 0 \\ -v^T & 1 \end{bmatrix} = \begin{bmatrix} I + \frac{uv^T}{1-v^Tu} & -\frac{u}{1-v^Tu} \\ -\frac{v^T}{I-v^Tu} & \frac{1}{I-v^Tu} \end{bmatrix}$.

From (5), we have $E^{-1} = \begin{bmatrix} (I-uv^T)^{-1} & 0 \\ -v^T(I-uv^T)^{-1} & 1 \end{bmatrix} \begin{bmatrix} I & -u \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (I-uv^T)^{-1} & -\frac{u}{I-uv^T} \\ -\frac{v^T}{I-v^Tu} & \frac{uv^T}{I-uv^T} \end{bmatrix}$. By comparing the (1,1) blocks, we can see that $(I-uv^T)^{-1} = I + \frac{uv^T}{1-v^Tu}$.

III.1 #3 $M^{-1} = (A-uv^T)^{-1} = A^{-1} + A^{-1}u(I - v^TA^{-1}u)^{-1}v^TA^{-1}$.

For the example, $(A-uv^T)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. $A^{-1} + A^{-1}u(I - v^TA^{-1}u)^{-1}v^TA^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1 - \frac{1}{3})^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. So the formula is true.

III.1 #4 $(A-uv^T)y = (A-uv^T)(x + \frac{v^Tx}{1-v^Tz}z) = Ax + \frac{Av^Txz}{1-v^Tz} - uv^Tx - \frac{uv^Tv^Txz}{1-v^Tz} = b + \frac{Av^Txz}{1-v^Tz} - \frac{uv^Tx - uv^Txv^Tz + uv^Tv^Txz}{1-v^Tz} = b + \frac{(Az-u)v^Tx}{1-v^Tz} = b$.

III.1 #5 $M^{-1}M = (A - UV^T)A^{-1} + (A - UV^T)(A^{-1}U(I - V^TA^{-1}U)^{-1}V^TA^{-1}) = I - UV^TA^{-1} + U(I - V^TA^{-1}U)(I - V^TA^{-1}U)^{-1}V^TA^{-1} = I - UV^TA^{-1} + UV^TA^{-1} = I$.

III.1 #6 This would give us: $(A-I)^{-1} = A^{-1} + A^{-1}(I - A^{-1})^{-1}A^{-1}$.

Then, we can compute: $(A-I)(A^{-1} + A^{-1}(I-A^{-1})^{-1}A^{-1}) = I + (I-A^{-1})^{-1}A^{-1} - A^{-1} - A^{-1}(I-A^{-1})^{-1}A^{-1} = I + (I-A^{-1})(I-A^{-1})^{-1}A^{-1} - A^{-1} = I + A^{-1} - A^{-1} = I$.

III.1 #7 We need to show: $x + Zw = M^{-1}b = y$. $M^{-1}b = A^{-1}b + A^{-1}U(I - V^TA^{-1}U)^{-1}V^TA^{-1}b = x + Z(I - V^TZ)^{-1}V^Tx = x + Z(I - V^TZ)^{-1}(I - V^TZ)w = x + Zw$. Thus, we can solve the equation without computing M^{-1}.

III.1 #8 $\frac{d}{dt}A^2(t) = \lim_{\Delta t \to 0} A^2(t) = \lim_{\Delta t \to 0} \frac{(A(t)+\Delta A)^2 - A^2(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{A \cdot \Delta A + \Delta A \cdot A + \Delta A^2}{\Delta t} = A\frac{dA}{dt} + \frac{dA}{dt}A$.

III.1 #9 $\frac{dA^{-1}}{dt} = \begin{bmatrix} 0 & -2t \\ 0 & 0 \end{bmatrix}$. $-A^{-1}\frac{dA}{dt}A^{-1} = \begin{bmatrix} -1 & t^2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -t^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2t \\ 0 & 0 \end{bmatrix}$.

III.1 #10

$\hat{x}_{new} = \frac{b_1 + \cdots + b_{1000}}{1000} = \frac{b_1 + \cdots + b_{999}}{999}\frac{999}{1000} + \frac{b_{1000}}{1000} = \frac{b_1 + \cdots + b_{999}}{999} - \frac{b_1 + \cdots + b_{999}}{999}\frac{1}{1000} + \frac{b_{1000}}{1000} = \frac{b_1 + \cdots + b_{999}}{999} + \frac{1}{1000}(b_{1000} - \frac{b_1 + \cdots + b_{999}}{999}) = \hat{x}_{old} + \frac{1}{1000}(b_{1000} - \hat{x}_{old})$.