MATH 309 HW#11

Solution

III.2 #1 Since $u^Tu = 1$, $\frac{d(u^Tu)}{dt} = 0$. $\frac{d(u^Tu)}{dt} = \frac{d}{dt}(u \cdot u) = u \cdot u' + u' \cdot u = 2u^Tu' = 0$.

III.2 #2 From interlacing, we know that $\alpha_1 \geq \lambda_1 \geq \alpha_2 \geq \lambda_2 \geq ... \geq \alpha_n \geq \lambda_n$.

III.2 #3 (a) Eigenvalues: $\lambda = t + 1 \pm \sqrt{t^2 + 2t + 2}$.

(b) When t=0, the eigenvalues are $\lambda = 1 \pm \sqrt{2}$. So the corresponding eigenvectors are $\begin{bmatrix} 1 \pm \sqrt{2} \\ 1 \end{bmatrix}$. $\frac{d\lambda}{dt} = 1 \pm \frac{1}{2}(t^2 + 2t + 2)^{-\frac{1}{2}}(2t + 2) = 1 \pm \frac{1}{\sqrt{2}}$.

Also, $y^T \frac{dA}{dt} x = \begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \pm \sqrt{2} \\ 1 \end{bmatrix} = 1 \pm \frac{1}{\sqrt{2}}$ and we get LHS=RHS.

(c) $A(t) - A(0) = \begin{bmatrix} t & t \\ t & t \end{bmatrix}$ is positive semi-definite because it's symmetric and all its leading pivots are non-negative. $\lambda_1(t) = t + 1 + \sqrt{t^2 + 2t + 2} > 1 + \sqrt{2}$. $1 + \sqrt{2} > 0 > t + 1 - \sqrt{(t+1)^2 + 1} > 1 - \sqrt{2}$. Thus, $\lambda_1(t) > \lambda_1(0) > \lambda_2(t) > \lambda_2(0)$.

III.2 #4 The first i eigenvectors to create an i-dimensional subspace.

III.2 #5 The first matrix has eigenvalues 3,1,0; the second matrix has eigenvalues $\frac{\sqrt{5}+3}{2}$, $\frac{-\sqrt{5}+3}{2}$; the third matrix has eigenvalues 1.

Thus, they interlace: $3 \geq \frac{\sqrt{5}+3}{2} \geq 1 \geq 1 \geq \frac{-\sqrt{5}+3}{2} \geq 0$.

III.2 #6 1) Since $\frac{d\lambda}{dt} = y^T \frac{dA}{dt} x$, when t=0, $\frac{d\lambda_k}{dt} = y^T S x$ where y^T is the k_{th} row of the n-by-n matrix $\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & 1 & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$, and x is the k_{th} column.

Thus, the derivative of the eigenvalues are the diagonal entries of S.

2) According to 1), $y^T S x$ will give us one leading principle minor of S, which is positive because S is positive definite. For t=0, $\lambda = n, \cdots, 2, 1$. For t that is small but positive, $\lambda_1 > n > \lambda_2 > \cdots > \lambda_n > 1$.

3) By Weyl's inequality, $n + t\lambda_{max}(A) \geq \lambda_i(D + tS) \geq 1 + t\lambda_{min}(A)$.

III.2 #7 1) A^*A is positive semidefinite, so the singular values of A is the same as the square root of the eigenvalues of A^*A. Let $A^*A = S$ in question 6. Then using the conclusion from question 6, we can know that $\frac{d\sigma_k}{dt} = \frac{d\sigma_k}{d\lambda_k}\frac{d\lambda_k}{dt} = \frac{1}{2\sqrt{\lambda}} y^T S x$.

2) Weyl's inequality tells us that $\sigma_{max} \leq \sqrt{t\lambda_1(A) + n}$, $\sigma_{min} \geq \sqrt{1 + \lambda_n(A)}$.

III.2 #8 (a) The sum of two dimensions of V and Z is $n + 1$, so V and Z must share a dimension. Thus, there should exist a combination of the victors in basis of Z to form at least one vector in the basis of V

(b) Let $z = \sum_{i=k}^{n} c_i q_i$. Then the Rayleigh quotient is $\frac{\sum_{k=i}^{n} c_k^2 \lambda_k}{c_k^2} \leq \lambda_i$. Thus, $\max(R(x), x \in V) \leq \lambda_i$. Since this is true for all V, we have $\min(\max(R(x), x \in V), V$ with dimension $i) \leq \lambda_i$. Choose a specific V that is spanned by $q_1, \cdots, q_i$. So $\min(R) \geq \lambda_i$. Thus, the max min characterization holds.

III.2 #9 From the Law of Inertia stated above the problem, the eigenvalues of S should have the same signs as the eigenvalues of LDL^T. Since D is a diagonal matrix, it displays its eigenvalues on its diagonals. Thus, the signs of pivots in D matches the signs of eigenvalues in S.

III.2 #10 To find the pivots in H, we do row operations: multiply $C^T S^{-1}$ to the first row, and we subtract it from the second row. Then, the second row becomes $\begin{bmatrix} 0 & -C^T S^{-1} C \end{bmatrix}$.

Thus, the first n pivot positions are from S, and the second n pivot positions are from $-C^T S^{-1} C$. It's also clear from above discussion that S will provide n positive eigenvalues since it is positive definite.

Since congruence preserves the signs of eigenvalues, the signs of eigenvalues of $C^T S^{-1} C$ should be the same as the signs of eigenvalues of S^{-1}.

Since S is a positive definite matrix, its inverse is also positive definite. So S^{-1} has positive eigenvalues.

Thus, $-C^T S^{-1} C$ has n negative eigenvalues, and it will result in n negative eigenvalues overall.