MATH309 HW12 Solution

III.3 #1 There is a typo in the original problem: the matrix B should come from equation (5) instead of equation (6).

Thus, according to equation (5), $B = E_{21} + E_{32} + ... + E_{n,n-1} - E_{1,n}$.

Then: $AK - KB = [0\ ...\ 0\ (A^n - I)b]$.

III.3 #2 Since $C = AH - HB$, we can consider the i, j entry of C: $C_{ij} = \frac{2i-1}{2} \cdot \frac{1}{i+j-1} + \frac{2j-1}{2} \cdot \frac{1}{i+j-1} = \frac{2i+2j-2}{2} \cdot \frac{1}{i+j-1} = 1$.

III.3 #3 Again, there is a typo in the original problem: it should be $AT - TA = C$.

$A = E_{2,1} + E_{3,2} + ... + E_{n,n-1}$

So $C = AT - TA = t_{-1}(E_{nn} - E_{11})$. We only have two nonzero entries: $(1,1)$ and (n,n). $C_{11} = -t_{-1}, C_{nn} = t_{-1}$, respectively.

III.3 #4 Since $H^*H = K^*K \cdot K^*K$, $\lambda_j(H^*H) = \lambda_j(K^*K \cdot K^*K) = \lambda_j(K^*K)^2$, which gives $\sigma_j(H) = |\sigma_j(K)|^2$.

III.3 #5 Let $AP - P(-A) = C$, $s1^T + 1s^T = D$. Then $C_{jk} = x_j \frac{s_j+s_k}{x_j+x_k} - (-x_k)\frac{s_j+s_k}{x_j+x_k} = s_j + s_k$. Also, $D_{jk} = s_j + s_k$. Thus, $AP - P(-A) = s1^T + 1s^T$.

III.3 #6 $AX - XB = C$, we multiply X^{-1} on both sides:

$X^{-1}(AX - XB)X^{-1} = X^{-1}CX^{-1}$.

Rearrange, we get: $X^{-1}A - BX^{-1} = X^{-1}CX^{-1}$.

III.3 #8 $AA^* = (S + Z) \cdot (S + Z)^* = SS^T + S\bar{Z}^T + ZS + Z\bar{Z}^T$.

$A^*A = (S+Z)^*(S+Z) = S^TS + S^TZ + \bar{Z}^TS + \bar{Z}^TZ = SS^T + SZ + S\bar{Z}^T - \bar{Z}Z = SS^T + ZS + S\bar{Z}^T + Z\bar{Z}^T$.

Thus, we get $A^*A = AA^*$, and A is normal.

III.5 #1 1) We need to minimize the eigenvalues of A_0^TA to minimize $||A||_N$. So, we need to let A have as low rank as possible. Thus, $A_0 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

2) To minimize B_0^TB, let $B_0 = \begin{pmatrix} 1 & x \\ y & 4 \end{pmatrix}$. Then $B_0^TB_0 = \begin{bmatrix} y^2+1 & x+4y \\ x+4y & x^2+16 \end{bmatrix}$. To minimize $x^2 + y^2, xy = 4$, we take $x = y = 2$. So $B_0 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

3) We need to make C_0 be a rank 1 matrix. Thus, the only way for C_0 to be rank 1 is $C_0 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

III.5 #2 If we multiply them together, we need $3a + 4b = 1$ and $3c + 4d = 0$. To minimize $||A||_S$, we want a, b, c, d to be all zero. In this case, $c = d = 0$ works, and the best minimal from $3a+4b = 1$ is $a = 0, b = 1/4$.