Math 309 HW3 Solution

1. **I.5, #1**

   $u+v$ is orthogonal to $u-v$:

   $(u+v)\cdot(u-v) = (u\cdot u) - (u\cdot v) + (u\cdot v) - (v\cdot v) = (u\cdot u) - (v\cdot v) = 1-1 = 0.$

   Length of $u+v$:

   $\|u+v\| = \sqrt{(u+v)\cdot(u+v)} = \sqrt{u\cdot u + u\cdot v + v\cdot u + v\cdot v} = \sqrt{1+0+0+1} = \sqrt{2}.$

   Similarly, the length of $u-v$ is:

   $\|u-v\| = \sqrt{(u-v)\cdot(u-v)} = \sqrt{u\cdot u - u\cdot v - v\cdot u + v\cdot v} = \sqrt{1-0-0+1} = \sqrt{2}.$

2. **I.5, #4**

   $(Qx)^T(Qy) = x^TQ^TQy = x^TIy = x^Ty = \|x\|\cdot\|y\|\cdot\cos\theta$, with θ as the angle in between. Thus, the length and angle are not changed.

3. **I.5, #5**

   For orthogonal Q, $Q^TQ = I$. Since the inverse of a matrix is unique, $Q^{-1} = Q^T$.

   Also, $(Q^{-1})^T = (Q^T)^T = Q = (Q^{-1})^{-1}$. Thus, Q^{-1} is orthogonal as well.

   Consider two orthogonal matrices Q_1, Q_2: $(Q_1Q_2)^T(Q_1Q_2) = Q_2^TQ_1^TQ_1Q_2 = Q_2^TIQ_2 = Q_2^TQ_2 = I$. Thus, the product of two orthogonal matrices are also orthogonal.

4. **I.5, #6**

   $P^TP = I$, so $P^{-1} = P^T$. All permutation matrices are orthogonal because they are just the identity matrix with row exchanges. If we multiply P^T with P, we are basically doing the row or column exchanges twice. Thus, we will get the identity matrix back.

5. **I.5, #7**

   $$PF = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i^2 & 0 \\ 0 & 0 & 0 & i^3 \end{pmatrix}$$

   So the eigenvalues of P are: $\lambda_1 = 1, \lambda_2 = i, \lambda_3 = i^2, \lambda_4 = i^3$.

   Let $Q = \frac{1}{2}F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{pmatrix}$.

   Thus, $\bar{Q}^TQ = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = I_4$. Thus, it has orthogonal columns.

6. **I.6, #1**

   $\lambda_1 + \lambda_2 = 2\cos\theta = tr(Q)$.

   $\lambda_1 \cdot \lambda_2 = \cos^2\theta + \sin^2\theta = \det(Q)$.

   Check eigenvectors: $\bar{x}_1 \cdot x_2 = \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 0$, so they are orthogonal.

   Since transpose does not change the eigenvalue of a matrix, $Q^{-1} = Q^T$ will have the same eigenvalue as Q.

7. **I.6, #4**

   A and B are triangular matrices, so we can read the eigenvalues from the diagonal: $\lambda_1 = \lambda_2 = 1$ for both matrices.

   Eigenvalue of AB and BA : $(1 - \lambda)(3 - \lambda) = 2$, so $\lambda = 2 \pm \sqrt{3}$.

   Thus, eigenvalues of AB are not equal to eigenvalues of A times eigenvalues of B, but eigenvalues of AB are equal to eigenvalues of BA.

   (In general, AB and BA always have the same eigenvalues, since they will have the same characteristic polynomial.)

8. **I.6, #6**

   For matrix A: $\lambda_1 = 1, \lambda_2 = 0.4$, with eigenvectors $x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

   For matrix A^∞: $\lambda_1 = 1, \lambda_2 = 0$, with eigenvectors $x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

   For matrix A^{100}: $\lambda_1 = 1, \lambda_2 = 0.4^{100}$, with eigenvectors $x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

   Since 0.4^{100} is very close to 0, A^{100} is close to A^∞.

9. **I.6, #12**

   Since the matrix has rank 1, we know that $\lambda_1 = \lambda_2 = 0$. Since the trace is the sum of the eigenvalues, we know $\lambda_3 = 6$.

   After we get the eigenvalues, we can find eigenvectors by doing row reduction on $(A - \lambda I)$ and find the basis for its null space. The eigenvectors we get are: $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ for $\lambda_1 = \lambda_2 = 0$, and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for $\lambda_3 = 6$.

   (Eigenvectors can be different, but the vector space they span should be the same.)

10. **I.6, #15**

a) For the first A, we can read the eigenvalues from the diagonals: $\lambda_1 = 1, \lambda_2 = 3$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(If a matrix has a constant row sum, then $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ must be one of its eigenvector.)

Thus, A can be written as $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

For the second A, its rank is one, so we have $\lambda_1 = 0$ and $\lambda_2 = tr(A) = 4$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(If a 2-by-2 matrix has only repeated columns, then $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ must be one of its eigenvector.)

Thus, A can be written as: $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3/4 & -1/4 \\ 1/4 & 1/4 \end{pmatrix}$.

b) $A^3 = (X\Lambda X^{-1})(X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda I \Lambda I \Lambda X^{-1} = X\Lambda^3 X^{-1}$.

$A^{-1} = (X\Lambda X^{-1})^{-1} = X\Lambda^{-1}X^{-1}$.

11. **I.6, #23**

$A = X\Lambda X^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{2}$, so $A^{1/2} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

B does not have a real square root because it has a negative eigenvalue -1, so $\Lambda^{1/2}$ for B will have imaginary numbers.