MATH309 HW4 Solution

I.6 #25 We can multiply columns of X and rows of ΛX^{-1}, using columns-by-rows matrix multiplication. Thus, $A = X\Lambda X^{-1} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \end{bmatrix} = \begin{bmatrix} x_1\lambda_1 & \cdots & x_n\lambda_n \end{bmatrix} \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \end{bmatrix} =$ $\lambda_1 x_1 y_1^T + \cdots + \lambda_n x_n y_n^T$.

(Notice that in columns-by-rows multiplication, we get n rank-1 matrices, and we add them up.)

I.7 #1 a) Since $Sx = \lambda x$, $y^T Sx = y^T \lambda x$.

b) Since $Sy = \alpha y$, $y^T S^T = \alpha y^T$, so $y^T S^T x = y^T Sx = y^T \alpha x$.

c) Use the expression in a) to minus expression in b), $(\lambda - \alpha)y^T x = 0$. So $\lambda - \alpha = 0$, $\lambda = \alpha$. If $\lambda \neq \alpha$, then $y^T x = 0$.

d) Suppose $S = QDQ^T$, so $y^T Sx = y^T QDQ^T x$. Then, $U = y^T Q$, $V = Q^T x$.

We can write it out: $y^T Sx = UDV = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 d_1 v_1 + \cdots u_n d_n v_n$.

Similarly, $x^T Sy = x^T QDQ^T y = V^T DU^T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = u_1 d_1 v_1 + \cdots u_n d_n v_n$.

Thus, $y^T Sx = x^T Sy$.

I.7 #2 Compute determinant and trace:

$det(S_1) = 35 - 36 = -1$, so it doesn't have two positive eigenvalues.

$trace(S_2) = -6 = \lambda_1 + \lambda_2$, so it also doesn't have two positive eigenvalues.

$det(S_3) = 100 - 100 = 0 = \lambda_1\lambda_2$, it doesn't have two positive eigenvalues.

$det(S_4) = 101 - 100 = 1 = \lambda_1\lambda_2$, and $trace(S_4) = 102 = \lambda_1 + \lambda_2$. Thus, S_4 has two positive eigenvalues and is thus positive definite.

Example for $x^T S_1 x < 0$: Let $x = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$, so $x^T S_1 x = 180 - 180 - 180 + 175 = -5 < 0$.

I.7 #3

1) S: $\lambda_1\lambda_2 = 9 - b^2 > 0$, so $-3 < b < 3$.

Thus, $S = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 9 - b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^T$.

2) S: $\lambda_1\lambda_2 = 2c - 16 > 0$, so $c > 8$.

Thus, $S = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & c - 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^T$.

3) S: $\lambda_1\lambda_2 = c^2 - b^2 > 0$, so $c^2 > b^2$. $S = \begin{bmatrix} 1 & 0 \\ \frac{b}{c} & 1 \end{bmatrix} \begin{bmatrix} c & b \\ 0 & c - \frac{b^2}{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{b}{c} & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & c - \frac{b^2}{c} \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{c} \\ 0 & 1 \end{bmatrix} =$ LDL^T.

(Notice that L needs to have 1s on its diagonal.)

I.7 #4 The eigenvectors of A may not be real. Thus, λ may not real.

I.7 #5 For S, $\lambda^2 - 6\lambda + 9 - 1 = 0$, so $\lambda_1 = 2$, with $x_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\lambda_2 = 4$, with $x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Thus, $S = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = 4 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

For B, using the same method, we can get $\lambda_1 = 25$, $\lambda_2 = 0$, $x_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$, $x_2 = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$.

Thus, we get $B = 25 \begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix}$.

I.7 #6 M is antisymmetric and also orthogonal. Since the trace of M is 0, and the eigenvalues can only be i and $-i$, the four eigenvalues of M must be $i, i, -i, -i$.

I.7 #7 For this matrix, the two eigenvalues are both 0.

Since $\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we get eigenvalue $x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. It has only one eigenvector, so it's eigenspace is a line.

I.7 #8 The other eigenvalue for the original matrix is $1 + 10^{-15}$, so the corresponding eigenvector would be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since the two eigenvector of this matrix is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the angle between the two eigenvectors is of 45 degrees.

I.7 #9 a) $S^2 = S^T S = S^{-1} S = I$.

b)Possible eigenvalues of S should only be 1 or -1. If not, it can not be both symmetric and orthogonal. Thus, all possible Λ of S should have 1 or -1 on its diagonal.