MATH 309 HW5 Solutions

I.7 #12 From the quadratic equation, we get $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. For S, we have $\lambda_1 = 1$, $\lambda_2 = -1$.

I.7 #14 $4(x_1 - x_2 + 2x_3)^2 = 4x_1^2 - 4x_1x_2 + 8x_1x_3 - 4x_1x_2 + 4x_2^2 - 8x_2x_3 + 8x_1x_3 - 8x_2x_3 + 16x_3^2$, so $S = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$.

We can rewrite S: $S = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \end{bmatrix}$, so the only pivot of S is 4, and rank(S)=1. $det(S) = 4(4 \cdot 16 - 64) - 4(-64 + 64) + 8(32 - 32) = 0$, and the eigenvalues are 24, 0 and 0.

I.7 #15 $det(2) = 2$, $det(\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}) = 10 - 4 = 6$, $det(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}) = 2 \cdot 31 - 2 \cdot 16 + 0 = 30$. $\frac{6}{2} = 3$, which is exactly equal to the second pivot. $\frac{30}{6} = 5$, which is exactly equal to the third pivot.

I.7 #16 S is positive definite when $c > 0$, $c^2 - 1 > 0$ and $c(c^2 - 1) - (c - 1) + 1 - c > 0$. Thus, $c > 1$.

T is positive definite when $d - 4 > 0$ and $5d - 16 - 2(10 - 12) + 3(8 - 3d) = -4d + 12 > 0$. Since there's no d satisfying both condition, T is never positive definite.

I.7 #22 For the first matrix, it's obvious that A should be in the form $\begin{bmatrix} 3 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$, then $a^2 + c^2 = 1$, $ab + cd = 2$, $b^2 + d^2 = 8$, which means that a=1, b=2, c=0, d=2. Thus, $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

Similarly, for the second matrix, A should be in the form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$, then $a^2 + c^2 = 1$, $ab + cd = 1$, $b^2 + d^2 = 6$, which means that a=1, d=$\sqrt{5}$, c=0, b=1. Thus, the corresponding A is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$.

I.7 #26 $det(S) = det(QDQ^T) = det(Q)det(D)det(Q^T) = 2 \cdot 5 \cdot det(Q)det(Q^T) = 2 \cdot 5 \cdot det(QQ^T) = 10$.

The eigenvalues are 2 and 5. The corresponding eigenvectors are $\begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$ and $\begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$. All the eigenvalues are positive, so S is positive definite.

I.7 #28 The eigen-values for this matrix should be $\lambda_n - \lambda_1, \cdots, \lambda_1 - \lambda_1$. Since λ_1 is the largest among all λs, these eigenvalues are all non-negative. Thus, $\lambda_1 I - S$ is positive semi-definite.
(b)$\lambda_1 I \geq x^T Sx$, so $x^T \lambda_1 x \geq x^T Sx$.
(c) $\lambda_1 \geq \frac{x^T Sx}{x^T x}$, we can maximize the right hand side by taking $x = x_1$, where x_1 is the corresponding eigen vector of λ_1. Thus, $\lambda_1 = \frac{x^T Sx}{x^T x}$ is the maximum.