MATH309 HW6 Solution

I.8 #2 Since $\lambda_1 \geq \cdots \geq \lambda_n$, $\lambda_1 c_1^2 + \cdots + \lambda_n c_n^2 \leq \lambda_1(c_1^2 + \cdots + c_n^2)$. So $R(x) \leq \lambda_1$. Equality holds when $c_1 = 1$, $c_2 = \cdots = c_n = 0$, which means $x = v_1$.

I.8 #3 $x^T v_1 = (c_1 v_1^T + \cdots + c_n v_n^T)v_1 = c_1$ (since eigenvectors are orthonormal). Thus $c_1 = 0$. Then for the same reason as #2, $R(x) \leq \lambda_2$ and the equality holds when $c_2 = 1$, $c_3 = \cdots = c_n = 0$.

I.8 #4 The maximum of $R(x) = x^T Sx/x^T x$ is λ_3, with the conditions $x^T v_1 = 0$, $x^T v_2 = 0$.

I.8 #5 $A = U\Sigma V^T$, then $A^T = (U\Sigma V^T)^T V\Sigma U^T$, so the singular values don't change. But $||Ax||$ is not necessarily equal to $||A^T x||$. A counter example: Let A $= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. $||A \begin{bmatrix} 1 \\ 0 \end{bmatrix} || \neq ||A^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} ||$.

I.8 #7 This problem doesn't specify which norm to use. If it's the spectral norm, the new norm would be σ_2. If it's the Frobenius norm, then the new value is $(\sigma_2^2 + ... + \sigma_n^2)^{1/2}$.

The new singular values are $\sigma_2, \cdots, \sigma_n$, and the new rank is $n - 1$.

I.8 #15 For a 3-by-3 matrix, Σ will have three parameters. In order to have linear independence in columns of U and V, their first columns have 2 parameters each, and their second columns have 1 parameter each, and their third columns have no free parameter. Thus, 2+1+0=3, and the matrix A has a total of 9 parameters.

For a 4-by-4 matrix, Σ will have 4 parameters. Use similar technique in 3-by-3, U and V will have 3+2+1+0=6 parameters each. Thus, it results in 4+6+6=16 parameters.

I.8 #17 $A^T Av = \lambda v$, $AA^T Av = \lambda Av$, so Av is a eigenvector.

I.8 #20 $\begin{bmatrix} -2 & -6 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. However, $\begin{bmatrix} -2 & -6 \\ 6 & 2 \end{bmatrix}^2 = \begin{bmatrix} -32 & 0 \\ 0 & -32 \end{bmatrix}$, but $\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 8^2 & 0 \\ 0 & 4^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -24 & -40 \\ 40 & 24 \end{bmatrix}$. So the eigenvalues and eigenvectors are different.

I.8 #21 $AA^T A = U\Sigma V^T V\Sigma U^T U\Sigma V^T = U\Sigma^3 V^T$. So the singular values are $\sigma_1^2, \cdots \sigma_r^3$.