MATH 309 HW7 Solution

I.9 #1 If we arrange the singular values of A with rank r as: $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r$, then the new singular values are σ_{k+1}, σ_{k+2}, ..., σ_r.

I.9 #2

1) Since $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the best rank-1 approximation is $3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2) Similarly, we can find $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$.

3) $A = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$.

I.9 #4 The original $A - A_1$ has maximum absolute row sum 3, so we want to find something smaller than that. Consider $A_2 = \begin{bmatrix} 1 & \frac{3}{4} \\ 4 & 3 \end{bmatrix}$: it is closer to A, because the maximum absolute row sum of $A - A_2$ is 2.75.

I.9 #5 Take an example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $QA = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}$, which has a different maximum absolute row sum than A.

I.9 #6 Since S is a positive definite matrix, S has n positive eigenvalues if it is an n-y-n matrix, and it is symmetric.

Thus, $S = Q\Sigma Q^T = \lambda_1 q_1 q_1^T + \cdots + \lambda_r q_r q_r^T$ is a singular value decomposition and also a spectral decomposition. Since $\lambda_1 q_1 q_1^T$ is the best rank one matrix approximation of S, $\lambda_1 q_1 q_1^T$ is the best rank one approximation of the L^2 matrix norm.

I.9 #8 $||A - A_1||_2 = \sigma_2$, $||A - A_2||_2 = \sigma_3$, so matrices with $\sigma_2 = \sigma_3$ will have $||A - A_1||_2 = ||A - A_2||_2$.

I.10 #1 $H = \begin{bmatrix} 5 & 2 \\ 2 & \frac{5}{4} \end{bmatrix}$.

$\det(S - \lambda M) = 0$ gives two eigenvalues: $\lambda = \frac{25 \pm \sqrt{481}}{8}$ and two eigenvectors: $x = \begin{bmatrix} \frac{\sqrt{481} \pm 15}{8} \\ 1 \end{bmatrix}$.

$\det(H - \lambda I) = 0$ gives the same eigenvalues $\lambda = \frac{25 \pm \sqrt{481}}{8}$, and $y = \begin{bmatrix} \frac{15 \pm \sqrt{481}}{16} \\ 1 \end{bmatrix}$

Verify: $x_1^T x_2$ is not zero, but $x_1^T M x_2 = 0$ and $y_1^T y_2 = 0$.

I.10 #2 $R(y) = \frac{5c^2 + 16cd + 20d^2}{c^2 + d^2}$, so $\frac{dR}{dc} = \frac{2d(-8c^2 - 15cd + 8d^2)}{(c^2 + d^2)^2}$, $\frac{dR}{dd} = \frac{2c(8c^2 + 15cd - 8d^2)}{(c^2 + d^2)^2}$.

Using this, we can derive that its maximum is $\frac{1}{2}(25 + \sqrt{481})$, and its minimum is $\frac{1}{2}(25 - \sqrt{481})$.

$R^*(x) = \frac{5a^2 + 8ab + 5b^2}{a^2 + 4b^2}$. $\frac{dR}{da} = \frac{-8a^2 b + 30ab^2 + 32b^3}{(a^2 + 4b^2)^2}$, $\frac{dR}{db} = \frac{2a(4a^2 - 15ab - 16b^2)}{(a^2 + 4b^2)^2}$.

Thus, the maximum is $\frac{25+\sqrt{481}}{8}$, and its minimum is $\frac{25-\sqrt{481}}{8}$. These maxima occur at the eigenvectors derived in #1.

I.10 #3 Since max $\frac{y^T Hy}{y^T y} = \lambda_1$, $Hy_1 = \lambda_1 y_1$. The maximum value for $\frac{x^T Sx}{x^T Mx}$ is also λ_1, so $Sx_1 = \lambda_1 Mx_1$, for $x_1 = M^{-1/2} y_1$. $Wlog$, we can get $x_1 = M^{-1/2} y_1$ and $x_2 = M^{-1/2} y_2$.