42/40.

MATH 309 HW8

Sample solution based on that of Fangming Xu

5 **I.10 #4** $det(S-\lambda M)=0$, so $(25-5\lambda)-16=0$. So $\lambda_1=\frac{9}{5}$. The corresponding eigen-vector is $\begin{bmatrix}1\\-\frac{4}{5}\end{bmatrix}$. In the other case, $\lambda_2=\infty$. Thus, the eigen-vector should be $\begin{bmatrix}0\\1\end{bmatrix}$.

5 **I.10 #6** (a) Since S is symmetric, $S=S^T$, so $Z^TSZ=Z^TS^TZ=(Z^TSZ)^T$. So it preserves the symmetry.
(b) Since S is positive definite, we can write $Z^TSZ=Z^TUDU^TZ=YDY^T$. So $x^TZ^TSZx=x^TYDY^Tx=X^T(\lambda_1 y_1^2+\cdots+\lambda_n y_n^2)x>0$. So Z^TSZ is also positive definite. Zx is not the zero vector because Z is invertible, so it must have rank n.

you can do it in a easier way: congruence preserves signs of eigenvalues, and $\lambda > 0$ means P.D.

5 **I.10 #7** Positive definite matrices are congruent to the identity matrix. $x^TAx=x^TZ^TIZx=x^Tz^TZx=Y^TY>0$, so A has positive eigen values and A is positive definite.

3 **I.10 #8** $S=uu^T$ so S is positive definite. Thus we can write S as $S=S^{\frac{1}{2}}S^{\frac{1}{2}}$ where $S^{\frac{1}{2}}=U\Sigma^{\frac{1}{2}}U^T$. Since we need to solve $M^{-1}Sx=\lambda x$, now we can let $v=S^{\frac{1}{2}}x$ and solve $S^{\frac{1}{2}}M^{-1}S^{\frac{1}{2}}v=\lambda S^{\frac{1}{2}}x=\lambda v$, which is basically the eigenvalue and eigen vector of $S^{\frac{1}{2}}M^{-1}S^{\frac{1}{2}}$. Then, just choose the minimum eigenvalue and the corresponding eigenvector.

→ consider I.10 #3...

5 **I.11 #1** $||v||_2^2=|v_1|^2+\cdots+|v_n|^2\le(|v_1|+\cdots+|v_n|)\cdot max(|v_j|)=||v||_1||v||_\infty$.

5 **I.11 #3** Since $(\sum|v_j|^2)\le n\cdot max\ \{|v_j|\}^2$, $||v||_2\le\sqrt{n}||v||_\infty$. For the second question, we know that $(\sum|v_j|)^2\le n\sum(|v_j|^2)$, (w = (1,1,1,... ,1).) So $||v||_1\le\sqrt{n}||v||_2$.

4 **I.11 #10** I have no idea what I am supposed to prove... Is it just $||A||_F^2=(tr(A^TA)^{\frac{1}{2}})^2=tr(A^TA)=(A,A)$. Yes, we do this by direct computations. Let A = [A_1 ..... A_n]. Then ||A||_F^2 = \sum_{i,j} |a_{ij}|^2 = \sum_{j=1}^n \sum_{i=1}^m |a_{ij}|^2 = \sum_j A_j^*A_j = tr(A*A).

5 **I.11 #12** For any ab_{ij}, we have $ab_{ij}=\sum_{k=1}^n a_{ik}\cdot b_{kj}\le n\cdot max\{entry\_in\_A\}\cdot max\{entry\_in\_B\}=n|||A|||_\infty|||B|||_\infty$. Thus, $|||AB|||_\infty\le n|||A|||_\infty|||B|||_\infty$. The next step is just multiplying both sides by $\sqrt{mp}$ and rearrange. $\sqrt{mp}|||AB|||_\infty\le\sqrt{mp\cdot n^2}|||A|||_\infty|||B|||_\infty$, so $\sqrt{mp}|||AB|||_\infty\le\sqrt{mn}|||A|||_\infty\sqrt{np}|||B|||_\infty$.

5 **Bonus**

(Forward:) $trace(A^TB)=\sum A_{ij}B_{ij}=||A||||B||=\sqrt{\sum|A_{ij}|^2}\cdot\sqrt{\sum|B_{ij}|^2}$, so $\sum|A_{ij}B_{ij}|\cdot\sum|A_{ij}B_{ij}|=\sum|A_{ij}|^2\cdot\sum|B_{ij}|^2$. Expanding both sides, we can see that while (A,B) is nonzero, B has to be a multiple of A for this equation to hold.

(Backward:) Write B = k A, while k is a non-zero constant. Thus, we can write (A,B) = $trace(A^TB)=\sum A_{ij}B_{ij}=\sum A_{ij}kA_{ij}=\sqrt{\sum|A_{ij}|^2}\cdot\sqrt{\sum k|A_{ij}|^2}=\sqrt{\sum|A_{ij}|^2}\cdot\sqrt{\sum|B_{ij}|^2}$.