Differential equations. Consider the system $Ax' = x$ with $x(0) = x_0$.

For one variable, $x' = ax$ with $x(0) = x_0$, we have $x = e^{at}x_0$.

Suppose $S^{-1}AS = D = \text{diag}(d_1, \ldots, d_n)$. Then

$$A = SDS^{-1} = d_1 S(:,1)S^{-1}(1,:) + \cdots + d_n S(:,n)S^{-1}(n,:).$$

For $y = Sx$, $Sx' = y'$ and $Dy' = y$ so that

$$y = e^{Dt}y_0 = \begin{bmatrix} e^{id_1 t} & & \\ & \ddots & \\ & & e^{id_n t} \end{bmatrix} y_0 \quad \text{and} \quad x = S^{-1}y = S^{-1}e^{Dt}Sx_0.$$

That is,

$$x = (e^{d_1 t}S(:,1)S^{-1}(1,:) + \cdots + e^{d_n t}S(:,n)S^{-1}(n,:))x_0.$$

$x_1'(t) = a_{11}x_1(t) + \cdots + a_{1n}x_n(t)$

$x_n'(t) = a_{n1}x_1(t) + \cdots + a_{nn}x_n(t)$

$y_1'(t) = \lambda_1 y_1(t)$

$y_n'(t) = \lambda_n y_n(t)$

Example: Solve $x' = Ax$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = SDS^{-1}$ with $D = \text{diag}(\lambda_1, \lambda_2) = \text{diag}((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ and

$$S = \begin{bmatrix} 1 & (\sqrt{5}-1)/2 \\ (\sqrt{5}-1)/2 & -1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \frac{2}{5-\sqrt{5}}S.$$

So,

$$x = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t} \\ e^{\lambda_1 t} - e^{\lambda_2 t} \end{bmatrix}.$$

$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

$x_1'(t) = x_1(t) + x_2(t)$

$x_2'(t) = x_1(t)$

In fact, one can solve the equation $x'' = x' + x$ with $x(0) = (1,0)^T$.

Now, $\begin{pmatrix} x' \\ x \end{pmatrix}' = A \begin{pmatrix} x' \\ x \end{pmatrix}'$ so that $x = \frac{1}{\sqrt{5}}e^{\lambda_1 t} - e^{\lambda_2 t}$.

Exercise Solve the system $x' = Ax$ with the above A and $x_0 = (2,1)^T$.

$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = S^{-1}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = S\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

Differential equation:

Solve

$$y_1'(t) = \lambda_1 y_1$$
$$y_2'(t) = \lambda_2 y_2$$
$$\vdots$$
$$y_n'(t) = \lambda_n y_n$$

given $y_j(0)$, $j=1,\dots,n$

Then

$$y_j(t) = \boxed{e^{\lambda_j t}\, y_j(0)}$$

$$x' = Ax + O(x^2), \quad A(t^2)x$$

$$x_1'(t) = a_{11}x_1(t) + \dots + a_{1n}x_n(t)$$
$$x_2'(t) = a_{21}x_1(t) + \dots + a_{2n}x_n(t)$$
$$\vdots$$
$$x_n'(t) = a_{n1}x_1(t) + \dots + a_{nn}x_n(t)$$

$$x'(t) = Ax(t)$$

$A = SDS^{-1}$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$$x'(t) = SDS^{-1}x(t)$$

$$\left(S^{-1}x'(t)\right) = D\left(S^{-1}x(t)\right)$$

$$y'(t) = Dy(t)$$

$$\begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

$y = S^{-1}x(t)$, $S^{-1} = $

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & & \\ b_{31} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$S\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$y_j'(t) = \left[b_{j1}x_1(t) + \dots + b_{jn}x_n(t)\right]'$$
$$= b_{j1}x_1'(t) + \dots + b_{jn}x_n'(t)$$

$$(y^n)' = a_1 y^{(n)} + a_2 y^{(n-1)} + \cdots + a_{n+1} y^{(0)}$$

$$\begin{pmatrix} (y^n) \\ (y^{(n-1)}) \\ \vdots \\ (y^{(0)}) \end{pmatrix}' = \underbrace{\begin{pmatrix} a_1 & a_2 & \cdots & a_{n+1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & \bigcirc \\ \bigcirc & & \ddots & \\ & & 1 & 0 \end{pmatrix}}_{(n+1)} \begin{pmatrix} y^{(n)} \\ \vdots \\ y^{(0)} \end{pmatrix}$$

$$= (S \underset{\phi}{D} S^{-1}) \begin{pmatrix} y^{(n)} \\ \vdots \\ y^{(0)} \end{pmatrix}$$

Solve

$$\begin{pmatrix} \boxed{\hat{y}(n)} \\ \vdots \\ \hat{y}(0) \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} \hat{y}^{(n)}(0) \\ \vdots \\ e^{\lambda_n t} \hat{y}^{(0)}(0) \end{pmatrix}$$

~~Then recover [illegible]~~

where

$$\begin{pmatrix} \hat{y}(n) \\ \vdots \\ \hat{y}(0) \end{pmatrix} = S^{-1} \begin{pmatrix} y(n) \\ \vdots \\ y(0) \end{pmatrix}$$

$$\begin{pmatrix} y(n) \\ \vdots \\ y(0) \end{pmatrix} = S \begin{pmatrix} \hat{y}(n) \\ \vdots \\ \hat{y}(0) \end{pmatrix}$$

1.7 Symmetric matrices, Hermitian matrices, positive definite matrices

Theorem (a) Every real symmetric matrix A can be written as QDQ^T for a real diagonal matrix $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and an orthogonal matrix Q.

(b) Every complex Hermitian matrix A can be written as UDU^* for a real diagonal matrix $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and a unitary matrix U.

Proof. If $A \in M_2(\mathbb{R})$ be symmetric, let $v = [v_1, v_2]^T \in \mathbb{R}^2$ be such that $v^T v = 1$ and

$$v^T A v = \lambda_1 = \max\{x^T A x : x \in \mathbb{R}^2, x^T x = 1\}.$$

Then $Q = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}$ is orthogonal, and we claim that $Q^T A Q = \mathrm{diag}(\lambda_1, \lambda_2)$. If not, assume $Q^T A Q = \begin{pmatrix} \lambda_1 & b \\ b & c \end{pmatrix}$. Then

$$f(\theta) = [\cos\theta, \sin\theta] Q^T A Q [\cos\theta, \sin\theta]^T = \cos^2\theta\lambda_1 + \cos\theta\sin\theta b + \sin^2\theta c,$$

and $f'(0) = 2b \neq 0$. So, we can find θ near 0 such that $u^T A u > \lambda_1$ with $u = Q[\cos\theta, \sin\theta]^T$, which is a contradiction.

Now, for $A \in M_n(\mathbb{R})$. Let $v \in \mathbb{R}^n$ be a unit vector such that

$$v^T A v = \lambda_1 = \max\{x^T A x : x \in \mathbb{R}^n, x^T x = 1\}.$$

Suppose Q is an orthogonal matrix such that $Q^T A Q = A_1$. Then $A_1 = [\lambda_1] \oplus A_2$ ....

For the complex case, we consider a unit vector $v \in \mathbb{C}^n$ such that

$$v^* A v = \lambda_1 = \max\{u^* A u : u \in \mathbb{C}^n, u^* u = 1\}.$$

Remark. $AS = S\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$, $S^{-1}AS = \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix} = D$

$A = SDS^{-1}$

Example $A = \begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}$

$0 = \det(\lambda I - A) = \det\begin{pmatrix}\lambda-2 & -3\\ 0 & \lambda-1\end{pmatrix} = (\lambda-2)(\lambda-1)$

$S = [v_1\ v_2]$ s.t.

$AS = [Av_1\ Av_2] = [v_1\ v_2]\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

$S^{-1}AS = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

If there were an orthogonal Q s.t.

$Q^TAQ = Q^{-1}AQ = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

$\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix} = A = Q\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}Q^T$

$\left(Q\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}Q^T\right)^T = (Q^T)^T\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}^T Q^T$

$= Q\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}Q^T$

Note for explaining the proof

① Let

$$V = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\sqrt{1^2+2^2+3^2}} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

⇓ apply Gram-Schmidt

Find V_2 V_3 s.t.

$V = V_1$

$[V_1\ V_2\ V_3]$ is orthogonal.

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② $Q^T A Q = \begin{bmatrix} \lambda_1 & 0 & 0 & c & * & \cdots & * \\ 0 & & & & & & \\ 0 & & & * & & & \\ c & & & & & & \\ * & & & & & & \\ \vdots & & & & & & \end{bmatrix}$ (jth) $\quad c \neq 0$

Consider $(\cos\theta\ \sin\theta)\begin{pmatrix} \lambda_1 & c \\ c & d \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \lambda_1 + \varepsilon > \lambda_1$

$$[\cos\theta\ 0 \cdots 0\ \sin\theta\ 0 \cdots 0]\, Q^T A Q \begin{bmatrix} \cos\theta \\ 0 \\ \vdots \\ 0 \\ \sin\theta \\ 0 \\ \vdots \\ 0 \end{bmatrix} = (\cos\theta\ \sin\theta)\begin{bmatrix} \lambda_1 & c \\ c & d \end{bmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

(jth)

$= \lambda_1 + \varepsilon$

unit

$$\|Qx\|^2 = x^T Q^T Q x = x^T I x = x^T x = \|x\|^2$$

Positive (semi)definite matrices.

Definition A real symmetric matrix $A \in M_n(\mathbb{R})$ is positive definite (semi-definite) if it has positive (nonnegative) eigenvalues

A complex Hermitian matrix $A \in M_n(\mathbb{C})$ is positive definite (semi-definite) if it has positive (nonnegative) eigenvalues

Equivalent conditions

1. $x^*Ax > 0$ for all nonzero vector x.
2. $S = A^*A$ for an invertible A.
3. A has positive leading principal minors meaning ...
4. All pivots of A are positive. [In the Gaussian elimination process.] Then $A = LDL^*$ or $A = \tilde{L}\tilde{L}^*$ for some invertible lower triangular matrix $\tilde{L}$, the Cholesky factorization.

Example. $A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$.

Applications 1. Maximum and minimum of real valued function $f(x_1, \ldots, x_n)$.

2. Major and minor axes of elliptical disk/ellipsoid.

Note:

If $\overset{(a_{ij})}{A} = S \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} S^{-1}$

then $\operatorname{trace}(A) = a_{11} + \cdots + a_{nn} = \lambda_1 + \cdots + \lambda_n$

$\det(A) = \lambda_1 \cdots \lambda_n$.

Proof:

$$\det(A) = \det(S)\,\underline{\det\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}\,\det(S^{-1})$$
$$= (\lambda_1 \cdots \lambda_n)\,\det(S)\det(S^{-1})$$
$$= (\lambda_1 \cdots \lambda_n)\,\det(SS^{-1})$$
$$= (\lambda_1 \cdots \lambda_n)\,\det(I_n)$$
$$= (\lambda_1 \cdots \lambda_n)$$

Note that

$$\operatorname{trace}\big((x_{ij})(y_{ij})\big) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}y_{ji}$$
$$= \operatorname{trace}\big((y_{ij})(x_{ij})\big)$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$
$$= x_{11}y_{11} + x_{12}y_{21} + x_{21}y_{12} + x_{22}y_{22}$$
$$= \sum_{i=1}^{2}\sum_{j=1}^{2} x_{ij}y_{ji}$$
$$= (y_{ij} \cdots)(x_{ij})$$

$\therefore$ $\overset{a_{11} + \cdots + a_{nn}}{=} \operatorname{trace} A$

$$= \operatorname{trace}\Big(\underbrace{S\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{X}\underbrace{S^{-1}}_{Y}\Big)$$
$$= \operatorname{trace}\Big(\underbrace{S^{-1}S}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\Big)$$
$$= \operatorname{trace}\Big(I\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\Big)$$
$$= \operatorname{trace}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$
$$= \lambda_1 + \cdots + \lambda_n$$

Given $A = A^T \in M_n(\mathbb{R})$ has all positive values

Then $A = Q \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} Q^T \quad \lambda_1, \dots, \lambda_n > 0$

$x^T A x = x^T Q \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} Q^T x$

$= [y_1 \cdots y_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2 > 0$

$y_1 \cdots y_n$ are not all zero

$\because \left\| \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\| = \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\| \neq 0$

$\therefore \quad x^T A x > 0 \qquad \forall x \neq 0$

If $x^T A x > 0 \qquad \forall x \neq 0$.

$\therefore$ A has all positive eigenvalues

Prove the contra-positive : Show. If A has a non-positive eigenvalue λ.

Let $x \neq 0$ be s.t. ~~$\|x\| = 1$~~

$Ax = \lambda x \qquad \lambda \leq 0$

Then $x^T A x = x^T \lambda x = \lambda \|x\|^2 \leq 0$

$\therefore \quad x^T A x \not> 0$