__Recall__ If $A = A^*$ then $U^*AU = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

$\lambda_1, \dots, \lambda_n \in \mathbb{R}$. $\quad U^*U = I_n$

In the real case, then $A = A^T$.
and there is an orthogonal Q s.t.

$$Q^TAQ$$

In our proof, we use $\overset{\|v\|=1}{v^*Av} = \max\{x^*Ax : \|x\| = 1\}$
and show that

$$U_1^*AU_1 = \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & A_1 \end{array}\right]$$

and we can induct on $A_1 \in M_{n-1}$.

__In practice.__ We find an eigenvector for the largest eigenvalue

Note. $A = A^*$, If $Ax = \lambda x$. then we can choose x
to be a unit vector, say, replace x by $x/\|x\|$ so that

$$x^*Ax = \cancel{\lambda A} \quad \cancel{x^*Ax} \quad x^*(\lambda x) = \lambda(x^*x) = \lambda.$$

So we have $\underbrace{}_{}$ unitary/orthogonal

$$\cancel{A[x \mid x_2 \cdots x_n] = [x \mid \cdots \mid x_n]\left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & A_1 \end{array}\right]}$$

$A = \begin{bmatrix} 3 & 1+i & i \\ 1-i & 2 & -i \\ -i & i & 1 \end{bmatrix}$

$$U^*AU = \left[\begin{array}{c|c} \lambda & A_{12} \\ \hline 0 & A_{22} \\ \vdots & \\ 0 & \end{array}\right]. \text{ for } U = \underbrace{[x \mid x_2 \cdots x_n]}_{\text{is unitary/orthogonal.}}$$

But $(U^*AU)^* = U^*A^*U = U^*AU$

$$\therefore \left[\begin{array}{c|c} \bar{\lambda} & 0 \\ \hline A_{12}^* & A_{22}^* \end{array}\right] = \left[\begin{array}{c|c} \lambda & A_{12} \\ \hline 0 & A_{22} \end{array}\right] \quad \therefore \lambda = \bar{\lambda}, \quad A_{12} = 0, \; A_{12}^* = 0, \; A_{22} = A_{22}^*$$

Then induct on A_{22}.

To prove (0) A is positive definite

i.e., $A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$

$UU^* = I, \ \lambda_1, \cdots, \lambda_n > 0$

is equivalent to (1) — (4)

Proof: $\boxed{(0) \Rightarrow (1).}$ If $A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$

then for any $x \neq 0$, $x^* A x = x^* U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^* x.$

Let $y = U^* x = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \neq 0$

So $x^* A x = [\bar{y}_1 \cdots \bar{y}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$= \lambda_1 |y_1|^2 + \cdots + \lambda_n |y_n|^2 > 0$

$\boxed{(1) \Rightarrow (3)}$ The 1×1 minor. $\det([a_{11}]) = [1\ 0 \cdots 0]\, A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a_{11} > 0$

Let $A = (a_{ij})$:

$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & 0 \\ 0 & 0 \end{bmatrix}\right).$

~~det~~ ~~$\mu_1 \cdot \mu_2$~~

$= \det\left(U_2 \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} U_2^*\right) = \mu_1 \mu_2$

If μ_1 or μ_2 is ~~negative~~ non-positive then. $\underline{(a_{ij})}$

say $\mu_1 \leq 0$,

for $x = [u_{11} u_{12}] \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \cdots & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{11} \\ \bar{u}_{12} \end{bmatrix} = \mu_1 \leq 0$!!!

Contradicting (1). $\therefore \mu_1, \mu_2 > 0$

& $\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) > 0$

For 3×3 principal minor,

Consider $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \end{bmatrix} A \begin{bmatrix} I_3 \\ \hline 0 \end{bmatrix}$

$= U_3 \begin{bmatrix} \hat{\mu}_1 & & 0 \\ & \hat{\mu}_2 & \\ 0 & & \hat{\mu}_3 \end{bmatrix} U_3^*$. If $\mu_i \leq 0$, for $i \in \{1, 2, 3\}$.

there is $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ s.t. $\|v\| = 1$

$$\begin{bmatrix} a_{11} & \cdot & a_{13} \\ \vdots & & \\ a_{31} & & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mu_i \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Then

$$[\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3] \begin{bmatrix} a_{11} & \cdot & a_{13} \\ & & \\ a_{31} & & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= [\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3] \begin{bmatrix} I_3 \mid 0 \end{bmatrix} A \begin{bmatrix} I_3 \\ \hline 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mu_i \leq 0$$

~~Then~~ In general, we show that any $k \times k$ principal submatrix of A has positive eigenvalues

~~(2) ⇒ (3)~~

$\lambda_i > 0$ $\quad$ $x^* A x > 0$ $\quad$ $A = B^* B$ $\quad$ B^2 $\quad$ $B = B^*$

(0) $\quad$ (1) $\quad$ (2) $\quad$ (3) $\det(A_k) > 0$

$\boxed{(4) \quad A = L D L^*}$

$\sim(0) \Rightarrow \sim(1)$

$\cancel{(1) \Rightarrow (0)}$

Done

$A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$

(0)

(4)

(1) $\quad x^* A x > 0 \quad \forall x \neq 0$

(3)

(2) $\quad A = \left(U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^* \right) \left(U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^* \right)$

$= B^* B \qquad B = B^*$

---

(0) ⇒ (1) Done!

(1) ⇒ (0) $\quad$ use $\sim(0) \Rightarrow \sim(1)$.
Choose eigenvector corresponding to $\lambda_i \le 0$

(0) ⇒ (2) $\quad A = \underbrace{U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^*}\ \underbrace{U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^*} = B^* B \qquad B = U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^*$

(2) ⇒ (1) $\quad A = B^* B \Rightarrow x^* A x = x^* B^* B x = y^* y > 0$
$\qquad y = Bx \neq 0$

(3) ⇒ (4) $\quad$ By LU factorization. If A has leading principal minors then $A = LU = L D \hat{L} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$\Delta \qquad U = (u_{ij}) \qquad \nabla$

$D = \begin{pmatrix} \frac{1}{u_{11}} & & & 0 \\ & \frac{1}{u_{22}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{u_{nn}} \end{pmatrix} = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

(3) ⇒ (4) (Cont'd)

Remain to show that

$$L = \hat{L}^* \hat{L}$$

$$\begin{bmatrix} \diagdown & 0 \\ & \end{bmatrix} = \begin{bmatrix} \diagdown & \\ 0 & \end{bmatrix}^*$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & \hat{l}_{12} & \hat{l}_{13} \\ 0 & 1 & \hat{l}_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}$$

$$= d_1 \begin{bmatrix} 1 \\ l_{21} \\ l_{31} \end{bmatrix} \begin{bmatrix} 1 & \hat{l}_{12} & \hat{l}_{13} \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \\ l_{32} \end{bmatrix} \begin{bmatrix} 0 & 1 & \hat{l}_{23} \end{bmatrix} + d_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & x & x \\ x & & \\ x & & \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & x \\ 0 & x & x \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & a_{12} & a_{13} \\ a_{12} & * & \\ a_{13} & & \end{bmatrix}$$

$\therefore\ l_{21} = \overline{\hat{l}_{12}}$, $l_{31} = \overline{\hat{l}_{13}}$.

Then $\quad d_2 \begin{bmatrix} 0 \\ 1 \\ l_{32} \end{bmatrix} \begin{bmatrix} 0 & 1 & \hat{l}_{23} \end{bmatrix} + d_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$$= A - d_1 \begin{bmatrix} 1 \\ l_{21} \\ l_{31} \end{bmatrix} \begin{bmatrix} 1 & \bar{l}_{21} & \bar{l}_{31} \end{bmatrix}$$ is Hermitian

$\therefore\ d_2 \begin{bmatrix} 1 \\ l_{32} \end{bmatrix} \begin{bmatrix} 1 & \hat{l}_{23} \end{bmatrix}$ is Hermitian.

Same argument for $n \times n$ matrix A

Finally, $(4) \Rightarrow (1)$.

Suppose $A = L \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} L^*$. ~~$d_i \Rightarrow$~~ $d_i > 0$

Then for $x \neq 0$ $x^* A x = x^* L \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} L^* x$

$= [\bar{y}_1 \cdots \bar{y}_n] \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = d_1|y_1|^2 + \cdots + d_n|y_n|^2$

> 0

because $y = L^* x \neq 0$ & $\det(L) = 1$

$\therefore$ $x^* A x > 0$ $\forall$ nonzero x.

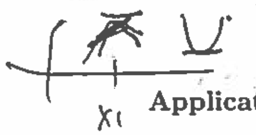
$\therefore$ We have established

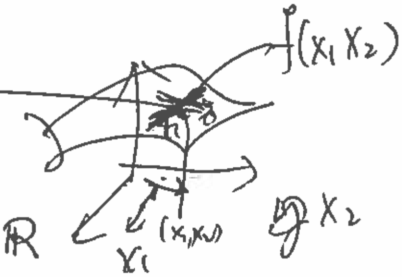
~~$(0)(2)(1)$~~. $(0) \Rightarrow (2) \Rightarrow (1) \Rightarrow (0)$

$\boxed{(1) \Rightarrow}$ $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$ $\boxtimes$

$f(x_1+t_1) = f(x_1) + t_1 f'(x_1) + \frac{t_1^2}{2!} f''(x_1) + O(t^3)$

$f(x_1, x_2)$





Applications

1. Maximum and minimum of real valued function $f(x) = f(x_1, \ldots, x_n)$. $\in \mathbb{R}$

Then $f(x_1 + t_1, \ldots, x_n + t_n)$

$$= f(x_1, \ldots, x_n) + \underbrace{(f_{x_1}(x), \ldots, f_{x_n}(x))(t_1, \ldots, t_n)^T}_{\text{Jacobian}} + (x_1, \ldots, x_n) J_f(x)(x_1, \ldots, x_n)^T + O(t^3),$$

where

$$J_f(x) = (f_{x_i,x_j}(x)) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right).$$

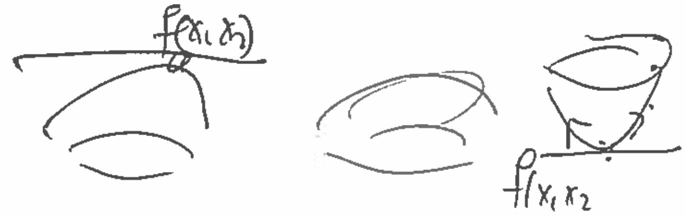
$x^T A x \geq 0$

$t_1 > 0$, < 0

Thus, $f(x)$ is minimum if $\boxed{f_{x_j}(x) = 0}$ for all $j = 1, \ldots, n$, and $\boxed{J_f(x) \text{ is}}$ positive semi-definite.

$\cancel{f(x_1+t_1) f(x_2}$ $\quad f(x_1+t_1, x_2+t_2) = f(x_1, x_2) + \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f(x)}{\partial x_2}\right)\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ $\quad > 0$, < 0

$$+ [t_1, t_2]\begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2}{\partial x_2 \partial x_2} \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} + O(t_1^3, t_2^3)$$



2. Major and minor axes of elliptical disk/ellipsoid.

Suppose the ellipse equation $1 = 5x^2 + 8xy + 5y^2$ is written as $1 = (x, y)A(x, y)^T$ with $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$. Then A is positive definite, and $A = QDQ^T$ with $D = \text{diag}(9, 1)$ and $Q = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then the ellipse equation becomes $1 = 9X^2 + Y^2$ with $X = (x + y)\sqrt{2}$ and $Y = (-x + y)/\sqrt{2}$. Geometrically, we apply a rotation of $-\pi/4$, we get a vertical ellipse.