Math 323 **Exam 1** **Name:** Sample Solution

1. Consider the LP: $\max Z = x_1 + 2x_2$

   Subject to: $-2x_1 + x_2 + x_3 \le 2$

   $-x_1 + x_2 - x_3 \le 1$

   $x_1, x_2, x_3 \ge 0.$

   (a) Show that the LP has no optimal solution (unbounded).

   (b) From the final simplex tableau, construct a feasible solution whose value of the objective function is larger than 2000.

) $\max Z = X_1 + 2X_2$

s.t. $-2X_1 + X_2 + X_3 + S_1 = 2$

$-X_1 + X_2 - X_3 + S_2 = 1$

$X_1, X_2, X_3, S_1, S_2 \ge 0$

	$(+1)X_1$	$(+2)X_2$	$(0)X_3$	$(0)S_1$	$(0)S_2$	constraint
	-2	1	1	1	0	2
	-1	1*	-1	0	1	1
	1	2	0	0	0	$Z=0$

B	$(+1)X_1$	$(+2)X_2$	$(0)X_3$	$(0)S_1$	$(0)S_2$	constraint
S_1	-1	0	2	1	-1	1
X_2	-1	1	-1	0	1	1
$\hat{c}$	3	0	2	0	-2	$Z=2.$

ecause $\hat{c}$ value of $X_1 = 3$, which is the most sitive value and the problem is a max. problem, ould choose X_1 as the entering variable. However, l of the coefficients under X_1 are negative, which eans increasing X_1 can keep increasing S_1 and X_2. ence, the LP is unbounded.

b) $-X_1 + X_2 = 1$ $\quad -X_1 + S_1 = 1$

$X_2 = 1 + X_1$ $\quad S_1 = 1 + X_1$

Let $X_1 = 700$, $X_2 = 701$, $X_3 = 0$

$Z = 700 + 1402 = 2102 > 2000$

$-2X_1 + X_2 + X_3 = -1400 + 701 = -699 \le 2$

$-X_1 + X_2 - X_3 = -700 + 701 = 1$, $X_1, X_2, X_3 \ge 0.$

$\therefore$ BV: $X_1 = 700$, $X_2 = 701$, $S_1 = 701$

NBV: $X_3 = S_2 = 0$

2. Below is the final tableau of the LP: $\max Z = 3x_1 + x_2 + 5x_3$ (in \$)

subject to: $6x_1 + 3x_2 + 5x_3 \le 45$ (labor)

$3x_1 + 4x_2 + 5x_3 \le 30$ (material)

$x_1, x_2, x_3 \ge 0$

C_B	X_B	$(3)x_1$	$(+1)x_2$	$(+5)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
3	x_1	1	$-1/3$	0	$1/3$	$-1/3$	5
5	x_3	0	1	1	$-1/5$	$2/5$	3
	$\bar{C}$	0	-3	0	0	-1	$Z = 30$

(a) Find the range of the unit profit c_1 associated with x_1 so that the basis $x_B = (x_1, x_3)$ remains to be optimal.

(b) Suppose an additional 15 units of material (in the original constraint) may be obtained at a cost of \$10. Is it profitable to do so? (Check if Z will increase more than \$10.)

a) $\hat{c} = c - c_B B^{-1} A$

$= [c_1\ \ 1\ \ 5\ \ 0\ \ 0] - [c_1\ \ 5]\begin{bmatrix} 1 & -\frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 1 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

$= [c_1\ \ 1\ \ 5\ \ 0\ \ 0] - [c_1\ \ -\frac{1}{3}c_1+5\ \ 5\ \ \frac{1}{3}c_1-1\ \ -\frac{1}{3}c_1+2]$

$= [0\ \ \frac{1}{3}c_1-4\ \ 0\ \ -\frac{1}{3}c_1+1\ \ \frac{1}{3}c_1-2]$

$\frac{1}{3}c_1 - 4 \le 0$, $\frac{1}{3}c_1 \le 4$, $c_1 \le 12$

$-\frac{1}{3}c_1 + 1 \le 0$, $\frac{1}{3}c_1 \ge 1$, $c_1 \ge 3$

$\frac{1}{3}c_1 - 2 \le 0$, $\frac{1}{3}c_1 \le 2$, $c_1 \le 6$.

$\therefore$ if $3 \le c_1 \le 6$, $X_B = (X_1, X_3)$ remains to be optimal.

b) $b = \begin{bmatrix} 45 \\ 45 \end{bmatrix}$ $B^{-1}b = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} 45 \\ 45 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$

$\therefore \begin{cases} X_1 = 0 \\ X_3 = 9 \end{cases} \Rightarrow Z = 3X_1 + X_2 + 5X_3 = 5 \times 9 = 45$

$\because 45 - 30 = 15 > 10$

$\therefore$ It is profitable.

3. The following tableau gives an optimal solution of a standard linear program:

$$\max Z = c^T x, \quad \text{subject to } Ax = b,\ x \geq 0.$$

C_B	X_B	$(2)x_1$	$(+3)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
2	x_1	1	0	1	3	-1	1
3	x_2	0	1	1	-1	2	2
	$\bar{C}$	0	0	-4	-3	-4	$Z=8$

(a) If $b = (b_1, b_2)^T$ is in the initial tableau, determine Δ so that $x_B = (x_1, x_2)$ will remain to be the optimal basis if $b = (b_1, b_2)^T$ is changed to $(b_1, b_2 + \Delta)^T$.

Note that $B^{-1}b$ will change to $B^{-1}(b_1, b_2+\Delta)^T = B^{-1}b + B^{-1}(0, \Delta)^T$.

(b) Find the optimal solution $(x_1, \ldots, x_5)$ and Z value of the problem if b_2 changes to $b_2 + 2$.

a) $\cancel{B^{-1}b = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2+\Delta \end{bmatrix} = \begin{bmatrix} 1-\Delta \\ 3+2\Delta \end{bmatrix}}$

$\cancel{1-\Delta \geq 0}$ $\quad$ $\cancel{3+2\Delta \geq 0}$

$\cancel{1 \geq \Delta}$ $\quad$ $\cancel{2\Delta \geq -3}$

$\cancel{\Delta \geq -\frac{3}{2}}$

$\therefore -\frac{3}{2} \leq \Delta \leq 1$

$$B^{-1}\begin{bmatrix} b_1 \\ b_2+\Delta \end{bmatrix} = B^{-1}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + B^{-1}\begin{bmatrix} 0 \\ \Delta \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ \Delta \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -\Delta \\ 2\Delta \end{bmatrix}$$

$$= \begin{bmatrix} 1-\Delta \\ 2+2\Delta \end{bmatrix}.$$

$1-\Delta \geq 0$ $\quad$ $2+2\Delta \geq 0$

$1 \geq \Delta$ $\quad$ $2\Delta \geq -2$

$\Delta \geq -1$

$\therefore -1 \leq \Delta \leq 1$

b) $B^{-1}\begin{bmatrix} b_1 \\ b_2+\Delta \end{bmatrix} = \begin{bmatrix} 1-2 \\ 2+2\times 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

C_B	B	$(2)X_1$	$(+3)X_2$	$(+1)X_3$	$(+0)X_4$	$(+0)X_5$	constraints
2	X_1	1	0	1	3	-1^*	-1
3	X_2	0	1	1	-1	2	~~8~~ 6
	$\hat{C}$	0	0	-4	-3	-4	~~Z=8~~

C_B	B	$(+2)X_1$	$(+3)X_2$	$(+1)X_3$	$(+0)X_4$	$(+0)X_5$	constraints
0	X_5	-1	0	-1	-3	1	1
3	X_2	2	1	3	5	0	4
	$\hat{C}$	-4	0	-8	-15	0	$Z=12$.

$\therefore X_1 = 0,\ X_2 = 4,\ X_3 = 0,\ X_4 = 0,\ X_5 = 1$

$Z = 12$

4. Consider the LP $\max Z = 3x_1 + 2x_2$

subject to:

$$-x_1 + 2x_2 \leq 4$$
$$3x_1 + 2x_2 \leq 14$$
$$x_1 - x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

(a) Use slack variables $s_1, s_2, s_3 \geq 0$ and convert the problem in standard form.

Determine the tableau for the basic feasible solution $(x_1, x_2, s_1, s_2, s_3) = (4, 1, 6, 0, 0)$.

Hint: $\begin{pmatrix} -1 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & -1 & 0 \end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & -3 \\ 5 & -1 & 8 \end{pmatrix}$.

(b) Find an optimal solution (y_1, y_2, y_3) and W value for the dual LP.

a) $\max Z = 3X_1 + 2X_2$

s.t. $-X_1 + 2X_2 + S_1 = 4$

$3X_1 + 2X_2 + S_2 = 14$

$X_1 - X_2 + S_3 = 3$

$X_1, X_2, S_1, S_2, S_3 \geq 0$

C_B	B	$(+3)X_1$	$(+2)X_2$	(0) ~~X_3~~ S_1	$(0)S_2$	$(0)S_3$	constraint
3	X_1	1	0	0	$\frac{1}{5}$	$\frac{2}{5}$	4
2	X_2	0	1	0	$1/5$	$\frac{-3}{5}$	1
0	S_1	0	0	1	$-1/5$	$\frac{8}{5}$	6
	$\hat{c}$	0	0	0	-1	0	$Z = 14$

$\because \hat{c} = C - C_B B^{-1} A \leq 0$

$\therefore (X_1, X_2, S_1, S_2, S_3) = (4, 1, 6, 0, 0)$ is an optimal solution of the primal LP.

b) $y = C_B B^{-1} = [3 \ \ 2 \ \ 0] \begin{bmatrix} 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{1}{5} & -\frac{3}{5} \\ 1 & -\frac{1}{5} & \frac{8}{5} \end{bmatrix}$

$= [0 \ \ 1 \ \ 0]$

$\therefore y_1 = 0, \ y_2 = 1, \ y_3 = 0$

$W = 4y_1 + 14y_2 + 3y_3 = 14.$

5. Consider the LP: $\min Z = x_1 + 4x_2 \quad + 3x_4$

Subject to:

$$x_1 + 2x_2 - x_3 + x_4 \geq 3$$

$$-2x_1 - x_2 + 4x_3 + x_4 \geq 2$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

Using excess variables $x_5, x_6 \geq 0$, we get the following tableau in the standard form:

C_B	B	$(1)x_1$	$(+4)x_2$	$(+0)x_3$	$(+3)x_4$	$(+0)x_5$	$(+0)x_6$	constraints
0	x_5	-1	-2	1	-1	1	0	-3
0	x_6	2	1	-4	-1	0	1	-2
	C	1	4	0	3	0	0	

Apply dual Simplex method to find the optimal solution $(x_1, \ldots, x_6)$ and Z.

$\because -3 < -2$

$\therefore$ choose ~~the~~ first row

$\frac{1}{-1} = -1 \quad \frac{4}{-2} = -2 \quad \frac{3}{-1} = -3$

$\because |-1| = 1$ is the minimum

$\therefore x_1$ ~~as~~ is the entering variable.

$\therefore B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

C_B	B	$(+1)X_1$	$(+4)X_2$	$(+0)X_3$	$(+3)X_4$	$(+0)X_5$	$(+0)X_6$	constraints
1	X_1	1	2	-1	1	-1	0	3
0	X_6	0	-3	-2^*	-3	2	1	-8
	$\hat{c}$	0	2	1	2	1	0	

C_B	B	$(+1)X_1$	$(+4)X_2$	$(+0)X_3$	$(+3)X_4$	$(+0)X_5$	$(+0)X_6$	constraints
1	X_1	1	$\frac{7}{2}$	0	$\frac{5}{2}$	-2	$-\frac{1}{2}$	7
0	X_3	0	$\frac{3}{2}$	1	$\frac{3}{2}$	-1	$-\frac{1}{2}$	4
	$\hat{c}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\frac{1}{2}$	$Z=7$

$\therefore (X_1, X_2, X_3, X_4, X_5, X_6) = (7, 0, 4, 0, 0, 0)$

$Z = 7$