Math 323 Operations Research **Notes on Chapter 7** C.K. Li

Transportation, Assignment, and Transshipment problem

7.1 Transportation problem

Example Powerco has 3 plants $P1, P2, P3$ that can supply powers 35, 50, 45 million of kilowatts, respectively, to 4 cities $C1, C2, C3, C4$ with demands of 45, 20, 30, 30 million of kilowatts. Shipping costs, demands, and supplies constraints are summarized in the following table.

	C1	C2	C3	C4	Supply
P1	\$8	\$6	\$10	\$9	≤ 35
P2	\$9	\$12	\$13	\$7	≤ 50
P3	\$14	\$9	\$16	\$5	≤ 40
Demand	≥ 45	≥ 20	≥ 30	≥ 30	

Suppose x_{ij} is the number of kilowatts from Pi to Cj, and let c_{ij} be the cost (in \$) from Pi to Cj.

The LP problem: $\min Z = \sum_{i,j} c_{ij}x_{ij}$

Subject to

$$\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &\leq 35 && \text{(P1 supply constraint)}\\
x_{21} + x_{22} + x_{23} + x_{24} &\leq 50 && \text{(P2 supply constraint)}\\
x_{31} + x_{32} + x_{33} + x_{34} &\leq 40 && \text{(P3 supply constraint)}\\
x_{11} + x_{21} + x_{31} &\geq 45 && \text{(C1 demand constraint)}\\
x_{12} + x_{22} + x_{32} &\geq 20 && \text{(C2 demand constraint)}\\
x_{13} + x_{23} + x_{33} &\geq 30 && \text{(C3 demand constraint)}\\
x_{14} + x_{24} + x_{34} &\geq 30 && \text{(C4 demand constraint)}\\
x_{ij} &\geq 0.
\end{aligned}$$

General Formulation Assume there are m suppliers shipping a certain product to n stores such that the cost of shipment from Si (ith supply point) to Dj (jth demand point) is c_{ij}. We need to solve the LP:

$$\max Z = \sum_{i,j} c_{ij}x_{ij}$$

subject to

$$\begin{aligned}
\textstyle\sum_j x_{ij} &\leq s_i && \text{for } i = 1, \dots, m, && \text{(supply constraints)}\\
\textstyle\sum_i x_{ij} &\geq d_j && \text{for } j = 1, \dots, n, && \text{(demand constraints)}\\
x_{ij} &\geq 0.
\end{aligned}$$

Remarks

1. The transportation company may want to solve the $\max Z = \sum_{i,j} c_{ij}x_{ij}$.
2. The problem is balanced if $\sum_i s_i = \sum_j d_j$.
3. If $\sum_i s_i > \sum_j d_j$, we may set up a dummy demand d_{n+1} with costs $c_{i,n+1} = 0$ for all i.
4. If $\sum_i s_i < \sum_j d_j$, we may set up a dummy supplier s_{m+1} with costs $c_{m+1,j}$ equals to the unit penalty amount imposed by Dj.

FIGURE 2
Graphical Representation of Unbalanced Powerco Problem and Its Optimal Solution (with Dummy Demand Point)

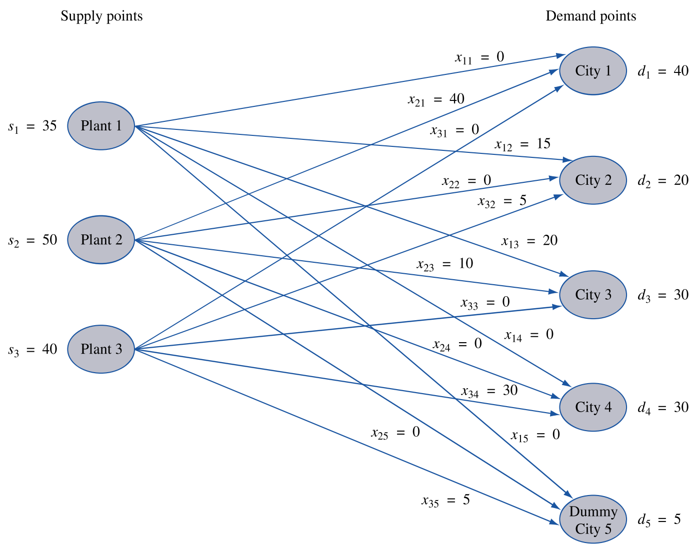

TABLE 2
A Transportation Tableau

					Supply
	c_{11}	c_{12}	⋯	c_{1n}	s_1
	c_{21}	c_{22}	⋯	c_{2n}	s_2
	⋮	⋮		⋮	⋮
	c_{m1}	c_{m2}	⋯	c_{mn}	s_m
Demand	d_1	d_2	⋯	d_n	

TABLE 3
Transportation Tableau for Powerco

	City 1	City 2	City 3	City 4	Supply
Plant 1	8	6 10	10 25	9	35
Plant 2	9 45	12	13 5	7	50
Plant 3	14	9 10	16	5 30	40
Demand	45	20	30	30	

Inventory problem as transportation problem

Example Sailco manufactures sailboats.

Demands for the next 4 quarters are: 40, 60, 75, 25.

At the beginning, there are 10 sailboats in inventory.

Each quarter, have to make 40 sailboats at the cost of $400 each.

Additional sailboat can be made at a cost of $450 each.

Left over inventory cost $20 per sailboat for each quarter.

We can formulate the following transportation problem.

Supply points.

S1 inventory (s1 = 10)
S2 quarter 1 regular production (s2= 40)
S3 quarter 1 overtime production (s3 = 150)
S4 quarter 2 regular production (s4= 40)
S5 quarter 2 overtime production (s5 = 150)
S6 quarter 3 regular production (s6= 40)
S7 quarter 3 overtime production (s7 = 150)
S8 quarter 4 regular production (s8= 40)
S9 quarter 4 overtime production (s9 = 150)

Here, regular production must be 40 per quarter. Overtime production has no limit, the total demand is 200, subtracting the initial inventory 10, and 40 regular production in the first quarter, so the maximum should be 150.

Demand points

D1 quarter 1 demand (d1 = 40)
D2 quarter 2 demand (d2 = 60)
D3 quarter 3 demand (d3 = 75)
D4 quarter 4 demand (d4 = 25)
D5 dummy demand (d5 = 700-200 = 570)

See the next page for the costs.

TABLE 6
Transportation Tableau for Sailco

	1	2	3	4	Dummy	Supply
Initial	[0] 10	[20]	[40]	[60]	[0]	10
Qtr 1 RT	[400] 30	[420] 10	[440]	[460]	[0]	40
Qtr 1 OT	[450]	[470]	[490]	[510]	[0] 150	150
Qtr 2 RT	[M]	[400] 40	[420]	[440]	[0]	40
Qtr 2 OT	[M]	[450] 10	[470]	[490]	[0] 140	150
Qtr 3 RT	[M]	[M]	[400] 40	[420]	[0]	40
Qtr 3 OT	[M]	[M]	[450] 35	[470]	[0] 115	150
Qtr 4 RT	[M]	[M]	[M]	[400] 25	[0] 15	40
Qtr 4 OT	[M]	[M]	[M]	[450]	[0] 150	150
Demand	40	60	75	25	570	

7.2 Finding a basic feasible solution

For a balanced transportation problem, there are mn variables x_{ij}, and $m+n-1$ linearly independent equalities.

1. To form a bfs, one needs to choose $m+n-1$ variables x_{ij}.
2. Arbitrary choices of $m+n-1$ variables may not correspond to a basic feasible solution.
3. The selection of those $\{x_{ij}\}$ do not contain a loop. That is, it contains a sequences

$$x_{i_1,j_1}, x_{i_1,j_2}, x_{i_2,j_2}, x_{i_2,j_3}, \ldots, x_{i_k,j_k}, x_{i_k,j_1}$$

so that $i_1, \ldots, i_k$ are distinct, and $j_1, \ldots, j_k$ are distinct.

Example $(s_1, s_2) = (4, 5), (d_1, d_2, d_3) = (3, 2, 4)$, $(x_{11}, x_{12}, x_{21}, x_{22})$ cannot form a bsf.

Reason. We have to solve

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 2 \\ 4 \end{bmatrix}.$$

Removing a redundant equality, we have to solve

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 4 \end{bmatrix}.$$

If $(x_{11}, x_{12}, x_{21}, x_{22})$ yields a bsf, then

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 4 \end{bmatrix},$$

which is impossible.

Three ways of finding basic feasible solutions

1. Northwest corner method.

   From the $(1, 1)$ entry, try to fulfill the row or column sum constraint in each step.

   Example: $(s_1, s_2, s_3) = (5, 1, 3), (d_1, d_2, d_3, d_4) = (2, 4, 2, 1)$.

2. Minimum cost method.

   Use the cheapest cost in each step to satisfy the row or column in each step.

   Example: $(s_1, s_2, s_3) = (5, 10, 15), (d_1, d_2, d_3, d_4) = (12, 8, 4, 6)$, $C = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 2 & 1 & 3 & 5 \\ 3 & 8 & 4 & 6 \end{pmatrix}$.

3. Vogel's method. Choose cheap costs and avoid future heavy penalty.

   Compute row/column penalties (difference of the two minimum costs in each row/column).

   Select basic variable at the row or column with maximum penalty.

TABLE 28

				Supply	Row Penalty
	[6]	[7]	[8]	10	7 – 6 = 1
	[15]	[80]	[78]	15	78 – 15 = 63
Demand	15	5	5		
Column Penalty	15 – 6 = 9	80 – 7 = 73	78 – 8 = 70		

TABLE 29

				Supply	Row Penalty
	[6]	[7] 5	[8]	5	8 – 6 = 2
	[15]	[80]	[78]	15	78 – 15 = 63
Demand	15	×	5		
Column Penalty	9	—	70		

TABLE 30

				Supply	Row Penalty
	[6]	[7] 5	[8] 5	0	—
	[15]	[80]	[78]	15	—
Demand	15	×	×		
Column Penalty	9	—	—		

TABLE 31

				Supply	Row Penalty
	[6] 0	[7] 5	[8] 5	×	—
	[15]	[80]	[78]	15	—
Demand	15	×	×		
Column Penalty	—	—	—		

7.3 The transportation simplex method

1. Set up the balanced transportation problem with m supply points and n demand points to minimize $Z = \sum_{i,j} c_{ij}x_{ij}$.

2. Find an initial basic feasible solution.

3. Find $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ with $u_1 = 0$ and $u_i + v_j = c_{ij}$ for those c_{ij} corresponding to the basic variables x_{ij}.

   Note that $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ is a "proposed" solution of the dual LP problem:

$$\max W = \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \qquad \text{Subject to } A^t \begin{bmatrix} u \\ v \end{bmatrix} \leq \begin{bmatrix} c_{11} \\ \vdots \\ c_{mn} \end{bmatrix},$$

$$u = [u_1, \ldots, u_m]^t, \ v = [v_1, \ldots, v_n]^t \text{ have entries with unrestricted signs.}$$

4. If $u_i + v_j \leq c_{ij}$ for all (i, j) pairs, then $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ is dual feasible. So, we get an optimal solution.

5. Otherwise, choose the (i, j) pair such that $u_i + v_j - c_{ij} > 0$ is maximum to be the entering variable.

6. Find a (the) loop using x_{rs} in the basic feasible solutions together with x_{ij}, and use x_{ij} as entry 0 in the loop.

7. Find the maximum $\delta > 0$ to add to the the even entries x_{rs} in the loop, and subtract δ from the odd entries in the loop.

   [An odd entries x_{rs} in the loop that is reduced to 0 after the procedure is the basic variable changing into a non-basic variable (as x_{ij} becomes a basic variable).]

8. Go back to Step 3 until an optimal solution (both primal and dual feasible) is found.

Remark For the maximization problem $\max Z = \sum_{i,j} c_{ij}x_{ij}$, the dual problem is:

$$\min W = \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \qquad \text{Subject to } A^t \begin{bmatrix} u \\ v \end{bmatrix} \geq \begin{bmatrix} c_{11} \\ \vdots \\ c_{mn} \end{bmatrix},$$

$$u = [u_1, \ldots, u_m]^t, \ v = [v_1, \ldots, v_n]^t \text{ have entries with unrestricted signs.}$$

So, we modify (4), (5) to:

4' The current solution is optimal if the proposed solution $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ of the dual problem satisfies $u_i + v_j \geq c_{ij}$ for all (i, j).

5' Otherwise, find the (i, j) pair such that $c_{ij} - (u_i + v_j) > 0$ is maximum to be the entering variable.

Example Solve the Powerco problem.

	C1	C2	C3	C4	Supply
P1	\$8	\$6	\$10	\$9	≤ 35
P2	\$9	\$12	\$13	\$7	≤ 50
P3	\$14	\$9	\$16	\$5	≤ 40
Demand	≥ 45	≥ 20	≥ 30	≥ 30	

	City1	City2	City3	City4	Supply
Plant1	8	6	10	9	35
Plant2	9	12	13	7	50
Plant3	14	9	16	5	40
Demand	45	20	30	30	

$u_1 =$ $v_1 =$

$u_2 =$ $v_2 =$ $\Delta =$

$u_3 =$ $v_3 =$

$v_4 =$

	City1	City2	City3	City4	Supply
Plant1	8	6	10	9	35
Plant2	9	12	13	7	50
Plant3	14	9	16	5	40
Demand	45	20	30	30	

$u_1 =$ $v_1 =$

$u_2 =$ $v_2 =$ $\Delta =$

$u_3 =$ $v_3 =$

$v_4 =$

	City1	City2	City3	City4	Supply
Plant1	8	6	10	9	35
Plant2	9	12	13	7	50
Plant3	14	9	16	5	40
Demand	45	20	30	30	

$u_1 =$ $v_1 =$

$u_2 =$ $v_2 =$ $\Delta =$

$u_3 =$ $v_3 =$

$v_4 =$

	City1	City2	City3	City4	Supply
Plant1	8	6	10	9	35
Plant2	9	12	13	7	50
Plant3	14	9	16	5	40
Demand	45	20	30	30	

$u_1 =$ $v_1 =$

$u_2 =$ $v_2 =$ $\Delta =$

$u_3 =$ $v_3 =$

$v_4 =$