What is $1 + 11 + \cdots + 11\cdots 11$?
What are scientific facts?

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Based on a talk of Professor Yiu-Tung Poon, Iowa State University.

What is $1 + 11 + \cdots + 11\cdots11$?

Outline

1. State the problem

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2. Looking for the answer

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3. Pattern

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4. Conjecture

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5. Solution

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Direct Computation

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n	$1 + 11 + \cdots + \underbrace{11\cdots11}_{n \text{ terms}}$	$=$	s_n
1	1	$=$	1
2	$1 + 11$	$=$	12
3	$1 + 11 + 111$	$=$	123
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	$1 + 11 + \cdots + \underbrace{11\cdots11}_{9 \text{ terms}}$	$=$	123456789

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What about $n \ge 10$?

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What about $n \geq 10$?

10	$1 + 11 + \cdots + \underbrace{11\cdots11}_{10 \text{ terms}}$	$=$	1234567900
11	$1 + 11 + \cdots + \underbrace{11\cdots11}_{11 \text{ terms}}$	$=$	12345679011

Formula

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$$1 + 2 + 3 \cdots + n = \frac{n(n+1)}{2}$$

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Sum of Arithmetic Progression

$$a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = \frac{n(2a+(n-1)d)}{2}$$

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$$2 + 5 + 8 + \cdots + 2999 = \frac{1000(3001)}{2} = 1500500$$

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Note: $\underbrace{11 \cdots 11}_{n \text{ terms}} = \frac{1}{9}\left(\underbrace{99 \cdots 99}_{n \text{ terms}}\right) = \frac{1}{9}(10^n - 1)$

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$$s_n = 1 + 11 + \cdots + \underbrace{11 \cdots 11}_{n \text{ terms}}$$

$$= \frac{1}{9}\left((10-1) + (10^2 - 1) + \cdots + (10^n - 1)\right)$$

Formula

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$$s_n = 1 + 11 + \cdots + \underbrace{11\cdots11}_{n \text{ terms}}$$
$$= \frac{1}{9}\left((10-1) + (10^2-1) + \cdots + (10^n-1)\right)$$
$$= \frac{1}{9}\left((10 + 10^2 + \cdots + 10^n) - n\right)$$

Formula

Sum of Sum of Geometric Progression

$$
\begin{aligned}
s_n &= 1 + 11 + \cdots + \underbrace{11\cdots11}_{n \text{ terms}} \\
&= \frac{1}{9}\left((10-1) + (10^2-1) + \cdots + (10^n-1)\right) \\
&= \frac{1}{9}\left((10 + 10^2 + \cdots + 10^n) - n\right) \\
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\end{aligned}
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$$= \frac{1}{9}\left((10 + 10^2 + \cdots + 10^n) - n\right)$$
$$= \frac{1}{9}\left(\frac{10(10^n-1)}{9} - n\right)$$

Example

$s_{15} =$

Formula

Sum of Sum of Geometric Progression

$$
\begin{aligned}
s_n &= 1 + 11 + \cdots + \underbrace{11 \cdots 11}_{n \text{ terms}} \\
&= \frac{1}{9}\left((10-1) + (10^2-1) + \cdots + (10^n - 1)\right) \\
&= \frac{1}{9}\left((10 + 10^2 + \cdots + 10^n) - n\right) \\
&= \frac{1}{9}\left(\frac{10(10^n-1)}{9} - n\right)
\end{aligned}
$$

Example

$$
s_{15} = \frac{1}{9}\left(\frac{10(10^{15}-1)}{9} - 15\right)
$$

Formula

Sum of Sum of Geometric Progression

$$s_n = 1 + 11 + \cdots + \underbrace{11\cdots11}_{n\ \text{terms}}$$
$$= \frac{1}{9}\left((10-1) + (10^2-1) + \cdots + (10^n - 1)\right)$$
$$= \frac{1}{9}\left((10 + 10^2 + \cdots + 10^n) - n\right)$$
$$= \frac{1}{9}\left(\frac{10(10^n-1)}{9} - n\right)$$

Example

$$s_{15} = \frac{1}{9}\left(\frac{10(10^{15}-1)}{9} - 15\right) = 123456790123455$$

Formula

Sum of Sum of Geometric Progression

$$\begin{aligned} s_n &= 1 + 11 + \cdots + \underbrace{11 \cdots 11}_{n \text{ terms}} \\ &= \frac{1}{9}\left((10-1) + (10^2-1) + \cdots + (10^n-1)\right) \\ &= \frac{1}{9}\left((10 + 10^2 + \cdots + 10^n) - n\right) \\ &= \frac{1}{9}\left(\frac{10(10^n-1)}{9} - n\right) \end{aligned}$$

Example

$$s_{15} = \frac{1}{9}\left(\frac{10(10^{15}-1)}{9} - 15\right) = 123456790123455$$

$$s_{101} = \frac{1}{9}\left(\frac{10(10^{101}-1)}{9} - 101\right) = ?$$

Formula

Sum of Sum of Geometric Progression

$$s_n = 1 + 11 + \cdots + \underbrace{11\cdots11}_{n \text{ terms}}$$
$$= \frac{1}{9}\left((10-1) + (10^2-1) + \cdots + (10^n - 1)\right)$$
$$= \frac{1}{9}\left((10 + 10^2 + \cdots + 10^n) - n\right)$$
$$= \frac{1}{9}\left(\frac{10(10^n-1)}{9} - n\right)$$

Example

$$s_{15} = \frac{1}{9}\left(\frac{10(10^{15}-1)}{9} - 15\right) = 123456790123455$$
$$s_{101} = \frac{1}{9}\left(\frac{10(10^{101}-1)}{9} - 101\right) = ?$$ Can we specify all digits?

Direct Computation

A simpler example, s_{23}

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$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & 1 & 1 \\
 & & & & & & & & & & & & 1 & 1 & 1 \\
 & & & & & & & & & & & & \vdots & \vdots & \vdots \\
 & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\hline
 & & & & & & & & & & & & & 2 & 3 \\
 & & & & & & & & & & & & 2 & 2 & \\
 & & & & & & & & & & & 2 & 1 & & \\
 & & & & & & & & & & & \vdots & & & \\
+ & 1 & & & & & & & & & & & & & \\
\hline
 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 9 & 0 & 1 & 2 & 3 & 4 & 3
\end{array}
$$

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S_{23}

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The sum can be put into the following compact form:

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S_{23}

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$$
\begin{array}{rl}
 & 10305070911131517192123 \\
+ & 02040608101214161820 22 \\
\hline
 & 12345679012345679012343
\end{array}
$$

Direct Computation

S_{23}

The sum can be put into the following compact form:

$$\begin{array}{r} 10305070911131517192123 \\ +\quad 02040608101214161820 22 \\ \hline 12345679012345679012343 \end{array}$$

Pattern

Direct Computation

S_{23}

The sum can be put into the following compact form:

$$\begin{array}{rr} & 10305070911131517192123 \\ + & 020406081012141618202 2 \\ \hline & 12345679012345679012343 \end{array}$$

Pattern

Discover the pattern:

Direct Computation

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The sum can be put into the following compact form:

$$\begin{array}{r} 10305070911131517192123 \\ +\quad 0204060810121416182022 \\ \hline 12345679012345679012343 \end{array}$$

Pattern

Discover the pattern: 123456790 is repeated from the left of the answer.

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The sum can be put into the following compact form:

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Discover the pattern: 123456790 is repeated from the left of the answer.

Use the pattern:

S_{23}

The sum can be put into the following compact form:

$$\begin{array}{r} 10305070911131517192123 \\ +\ 02040608101214161820 22 \\ \hline 1234567901234567901 2343 \end{array}$$

Pattern

Discover the pattern: 123456790 is repeated from the left of the answer.

Use the pattern: Start the calculation from the right until 123456790 appears.

Use the pattern

s_{101}

Use the pattern

s_{101}

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & 1 & 1 \\
 & & & & & & & & & & & & 1 & 1 & 1 \\
 & & & & & & & & & & & & \vdots & \vdots & \vdots \\
 & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\hline
 & & & & & & & & & & & & 1 & 0 & 1 \\
 & & & & & & & & & & & 1 & 0 & 0 & \\
 & & & & & & & & & & & 9 & 9 & & \\
 & & & & & & & & & & 9 & 8 & & & \\
 & & & & & & & & & 9 & 7 & & & & \\
 & & & & & & & & 9 & 6 & & & & & \\
 & & & & & \vdots & & & & & & & & & \\
+ & 1 & & & & & & & & & & & & & \\
\hline
 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 7 & 9 & 0 & 0 & 1
\end{array}
$$

Use the pattern

s_{101}

Use the pattern

s_{101}

	1	083086089092095098101
	02	082085088091094097100
+	003	081084087090093096099
	123456790	123456790123456790 01

Use the pattern

s_{101}

	1	083086089092095098101
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s_{101}

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s_{101}

	1	083086089092095098101
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+	003	081084087090093096099
	123456790	12345679012345679001

$$s_{101} = \underbrace{123456790 \cdots 123456790}_{m \text{ terms}}01$$

Use the pattern

s_{101}

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+	003	081084087090093096099
	123456790	1234567901234567901

$$s_{101} = \underbrace{123456790 \cdots 123456790}_{m \text{ terms}}01$$

where $m = \dfrac{101 - 2}{9} = 11.$

Conjecture

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Suppose $s_n = 1 + 11 + \cdots + \underbrace{11\cdots11}_{n \text{ terms}} = \underbrace{123456790\cdots123456790}_{m \text{ terms}} x_k \cdots x_1$

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Suppose $s_n = 1 + 11 + \cdots + \underbrace{11 \cdots 11}_{n \text{ terms}} = \underbrace{123456790 \cdots 123456790}_{m \text{ terms}} x_k \cdots x_1$

$$\begin{array}{rl} & 123456790123456790 \cdots 123456790 \cdots x_k \cdots x_1 \\ + & 111111111111111111111111111111 \\ \hline & 123456790123456790 \cdots 123456790 \cdots \cdots \end{array}$$

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1. $s_n = 1 + 11 + \cdots + \underbrace{11\cdots 11}_{n \text{ terms}} = \underbrace{123456790\cdots 123456790}_{m \text{ terms}} x_k \cdots x_1$

2. Evaluate the digits $1 + 11 + \cdots + \underbrace{11\cdots 11}_{n \text{ terms}}$ from the right. If we reach $123456790 x_k \cdots x_1$, then we have $m = \dfrac{n-k}{9}$.

Suppose $s_n = 1 + 11 + \cdots + \underbrace{11\cdots 11}_{n \text{ terms}} = \underbrace{123456790\cdots 123456790}_{m \text{ terms}} x_k \cdots x_1$

$$\begin{array}{rl} & 123456790123456790\cdots 123456790\cdots x_k \cdots x_1 \\ + & 1111111111111111111111111111 \\ \hline & 123456790123456790\cdots 123456790\cdots\cdots \end{array}$$

So, $s_n = 1 + 11 + \cdots + \underbrace{11\cdots 11}_{n+1 \text{ terms}} = \underbrace{123456790\cdots 123456790}_{m \text{ terms}} y_{k+1} \cdots y_1$

Test the conjecture

Small n

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Examine the digits of $\frac{1}{9}\left(\frac{10(10^n-1)}{9}-n\right)$.

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How do we calculate the first k digits (from the right) of $1+11+\cdots+11\cdots11+\underbrace{11\cdots11}_{n\text{ terms}}$ efficiently?

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Calculate $n+(n-1)10+(n-2)100+\cdots$

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Examine the digits of $\frac{1}{9}\left(\frac{10(10^n-1)}{9}-n\right)$.

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Calculate $n+(n-1)10+(n-2)100+\cdots=\sum_{r=0}^{k-1}(n-r)10^r$

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Examine the digits of $\frac{1}{9}\left(\frac{10(10^n-1)}{9}-n\right)$.

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+	003	081084087090093096099
	123456790	1234567901234567901

Calculate $n+(n-1)10+(n-2)100+\cdots=\sum_{r=0}^{k-1}(n-r)10^r$

Counter-example

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Possible values for $x_k \cdots x_1$

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Note that the first k digits (from the right) of s_n depends only on the first k digits of n.

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Possible values for $x_k \cdots x_1$

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this would disprove the conjecture.

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999999996 & \dots \\
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+ \quad \vdots \qquad \vdots \qquad\qquad & \vdots \\
\hline
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because $\dfrac{10^9-10}{9} = 11111110.$

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Discussion

- What is the meaning of scientific facts/theory?
- If there is a quantity that one cannot observe/measure, does the quantity matter?
- Can we prove the existence of a quantity by calculation?
- Other thoughts?

Thank you for your attention!