The Art of Symmetry and Groups

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Abstract

This paper explores a connection between math and art: symmetry. We first start by defining symmetry and look at examples of the symmetries of regular polygons. Then, we show that all of the possible symmetries of an object form a group. Expanding these symmetries to a one-dimensional line, we find that there are only seven possible resulting patterns, called frieze groups. If we go into two-dimensions, we find that there are seventeen possible tiling patterns, called wallpaper groups.

1 Introduction

Math connects to all other fields of study such as chemistry, physics, writing, music, and art. Within art, there is math in architecture, perspective pieces, and patterns. It is clear that humans like symmetry. Symmetry makes the world easier to understand and digest when we can predict what will happen next. There are so many examples of symmetry used in designs across cultures around the world: Muslim tiles, Greek columns, Egyptian vases, etc. We will dive deeper into the world of patterns, more specifically the math behind symmetry.

2 Symmetry

In mathematics, symmetry is a type of invariance, meaning that the object remains unchanged after a set of operations.[1] Symmetry is a distance-preserving bijection. We can move one part of the object and the overall picture will still look the same.

2.1 Types of Symmetries

There are six types of symmetries.[2]
Type 1: Reflectional symmetry is when a line or plane goes through an object and splits it into two mirror halves. This is what most people picture when they think of 'symmetry'. (Figure 1)[3]

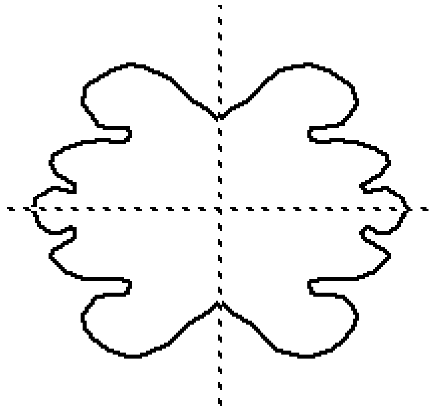

Figure 1: Refectional Symmetry

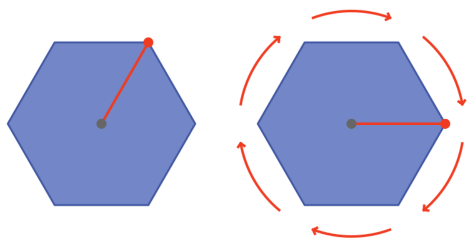

Figure 2: Rotational Symmetry

Type 2: Rotational symmetry is when an object can be rotated around a certain point or axis and not change the overall shape. (Figure 2)[4]

Type 3: Translational symmetry is when every point of an object is moved the same distance and still looks the same. (Figure 3)[5]

Type 4: Scale symmetry is when an object does not change after contracting or expanding. Fractals are an example of scale symmetry. Expanding or contracting a fractal results in a self-similar shape. (Figure 4)[6]

Type 5: Glide symmetry is a combination of a reflection and a translation. If this one is a

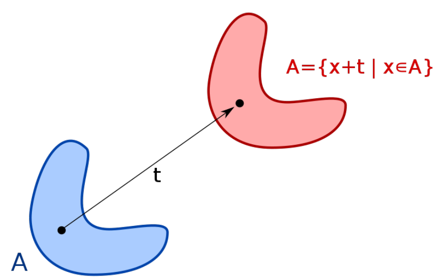

Figure 3: Translational Symmetry

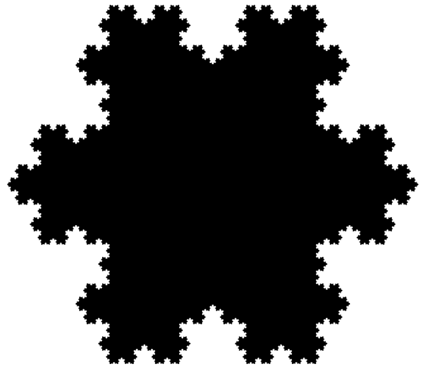

Figure 4: Scale Symmetry

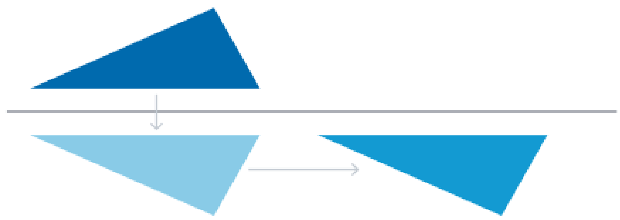

Figure 5: Glide Symmetry

little harder to visualize, you can think of footprints left behind after walking on sand. Your feet are reflections of each other, and one is translated with every step. (Figure 5)[7]

Type 6: Identity symmetry is doing nothing! Every object has identity symmetry.

Now, let us look at examples of symmetry in two-dimensional polygons. It should be noted that the identity is the same as making a full rotation. An isosceles triangle has the identity and 1 line of reflectional symmetry going through the incongruous side and the opposite vertex. An equilateral triangle has the identity, two other rotations, and three reflections. So, a regular 3-gon has six total symmetries. A square has the identity, three other rotations, and four reflections, giving it eight symmetries.

We can generalize this to any n-gon. Let the identity symmetry be denoted at e. The smallest increment of rotation for the n-gon is $\frac{2\pi}{n}$ radians, let this be r in the clockwise direction. Combining rotations gives us other symmetries such as $r^2, r^3, \ldots$ and we can continue rotating n times until we return to e. As for reflections, an n-gon has n axes reflections. However, we can align the n-gon vertically and find that all the axes can be found by combining a rotation r and a vertical reflection, f. If we were to vertically reflect twice, we would get e. We can combine rotations and vertical reflections:

$$\begin{aligned}&e, r, r^2, r^3, \ldots, r^{n-1}\\&f, rf, r^2f, r^3f, \ldots, r^{n-1}f\end{aligned}$$

We see that a regular n-gon has $2n$ symmetries. This is known as a dihedral group, denoted D_n, with n being the number of sides of the polygon. Now, let us define groups so the definition makes more sense.

3 Groups

Definition 1: A group $(G, +)$ is a non-empty set G with a binary operation "+" such that the following axiom are satisfied:

- Closure: $\forall a, b \in G, \text{we have } a + b \in G$
- Associativity: $\forall a, b, c \in G, \text{we have } (a + b) + c = a + (b + c)$
- Identity: $\exists e \in G, \text{such that } \forall a \in G \text{ we have } a + e = a$
- Inverse: $\forall a \in G, \exists -a$ (or a^{-1}) such that $a + (-a) = e$

An example of a group is integers under addition, $(\mathbb{Z}, +)$:

- Closure: An integer plus another integer is also an integer.
- Associativity: Addition is associative.
- Identity: $e = 0$, $\forall a \in \mathbb{Z}, a + 0 = a$
- Inverse: $\forall a \in G, \exists -a$ such that $a + (-a) = 0$

However, integers under multiplication, $(\mathbb{Z}, \cdot)$ is not a group:

- Closure: An integer times another integer is also an integer.
- Associativity: Multiplication is associative.
- Identity: $e = 1$, $\forall a \in \mathbb{Z}, a \cdot 1 = a$
- Inverse: Not every integer has a multiplicative inverse that is also an integer. For example, $2 \cdot \frac{1}{2} = 1 = e$, but $\frac{1}{2} \notin \mathbb{Z}$

We can look closer at D_n and see that it is a group:

- Closure: $r + r = r^2 \in D_n, r^2 + r = r^3 \in D_n, \ldots, r^{n-1} + r = r^n = e \in D_n$ Showing the combination of rotations and vertical reflections is closed is similar.
- Associativity: $(r + r^2) + r^3 = r^3 + r^3 + r^6 = r + r^4 = r + (r^2 + r^3)$ Generalizing this and combining vertical reflections is similar.
- Identity: e is the identity symmetry. $r + e = r$

(a) Tiles in the Alhambra de Granada in Spain (b) Fragment from the Erechtheion in Greece

Figure 6: Examples of friezes in art

- Inverse: Recall that r is a clockwise rotation. Then, $r + r^{n-1} = r^n = e, r^2 + r^{n-2} = r^n = e, \ldots, r^{n-1} + r = r^n = e$. For the reflections, repeating the same reflection gives you the original object. $f + f = e$

Symmetries that are their own inverses such as reflections and 180° rotations are called involutions.[8]

Theorem 1: Let G be a finite symmetry group with an even number of elements (like D_n). Then, G must have at least one involution.

Proof. Consider all of the symmetries. Since every element must have an inverse, we know there are pairs of inverses in G. Getting rid of the pairs, we are left with e and symmetries that do not have a matching pair. There are still an even number of elements left since we only got rid of pairs. Getting rid of e, there will be an odd number of symmetries left, and zero is not an odd number. There is no $k \in \mathbb{Z}$ such that $0 = 2k + 1$. Therefore, we are left with at least one involution. □

3.1 Frieze Groups

We know that the reflectional and rotational symmetries of an object form a group. If we place the object on a horizontal line, we can then translate and glide the object across the line. Using additional symmetries gives us more possibilities to create patterns in art. Because of the one-dimensional movement of the horizontal line, these patterns can be described as friezes. Friezes, examples in Figure 6, are a often used at decoration both two-dimensional and three-dimensional art.[9] As it turns out, there are only seven possible types of patterns you can create These groups are called frieze groups.[10] Descriptions and examples of the frieze groups can be found in Figure 7.[11]

3.2 Wallpaper Groups

What happens when we add a vertical line to move the object around? The resulting patterns would resemble tiling or wallpaper. You might think that having two directions of movement would give us fourteen different possibilities, but we actually get only seventeen possible wallpaper patterns, called wallpaper groups. This paper will not list all seventeen, but descriptions and examples of some of wallpaper groups can be found in Figure 8.[12]

It can be hard to differentiate between all the possible patterns, especially if you can not see it being made. If you are interested in creating a wallpaper pattern yourself, I would encourage you to check out an interactive wallpaper symmetry tool made by Professor David J. Eck.[13] Personally, I find it so interesting that every pattern we can think of can be boiled down to just seventeen all-inclusive categories. I will definitely be keeping a closer eye out for the types of symmetries and patterns on objects in my everyday life.

4 Presentation Comments

I wanted to connect math and art in some way that was not just architecture or geometry. While yes, symmetry is involved in both architecture and geometry, but I wanted to talk more about its place in art. I really enjoyed the discussion we had in class after this presentation. There were so many connections to other arts, such as music and writing, that I had no idea about until reading the discussion responses. We also talked about why people like symmetry so much. I feel like there is something relaxing about seeing a symmetrical repeating pattern, and something distressing when the pattern is suddenly broken. There were a few comments about how humans think symmetrical faces are more attractive. While I am not sure of the science behind it, I have heard that before and seen studies on it before.

Not everyone in this class has taken abstract algebra, so I wanted to approach the subject carefully by hopefully explaining everything clearly. It seems as though the ones who do not have a background in algebra were able to understand the group theory and appreciate the art behind it all.

Something I would have loved to talk about more in the presentation and this paper is M.C. Escher. I have seen his work on countless math slides and textbooks, and I would say that his work contributed to my interest in the connections between the two things I have dedicated my college career too: math and art.

Notes

[1]Wikipedia, "Symmetry in Mathematics".
[2]Wikipedia, "Symmetry".
[3]image from Wikipedia, "Reflection Symmetry".
[4]image from McBride, "GraphicMaths - Rotational Symmetry".
[5]image from Wikipedia, "Translational Symmetry".
[6]image from Roberts, "Fractals with R, Part 4: the Koch Snowflake".
[7]image from Wikipedia, "Glide Reflection".
[8]Birkbeck, "Inaugural Lecture by Professor Sarah Hart".
[9]image from The Sneaky Artist, "Escher and Insiration".
[10]image from Gardner, "Cast of a Fragment from the Erechtheion Frieze".
[11]images from Wikipedia, "Frieze Group".
[12]images from Wikipedia, "Wallpaper Group".
[13]Eck, "Wallpaper Symmetry".

5 References

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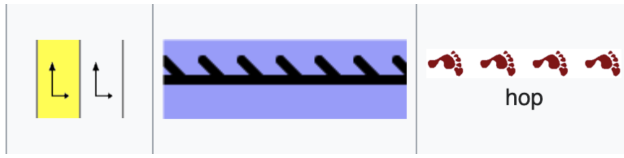
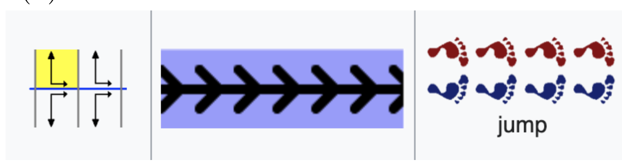
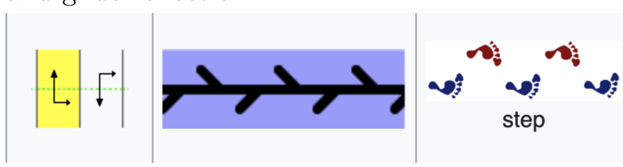


(a) T: Translation only

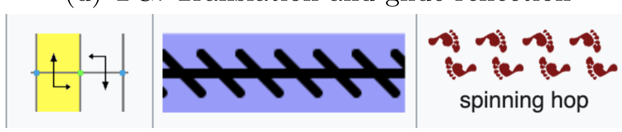
(b) TV: Translation and vertical line reflection



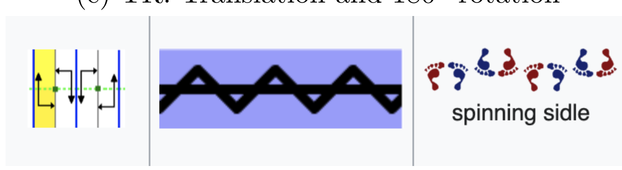
(c) THG: Translation, horizontal line reflection, and glide reflection



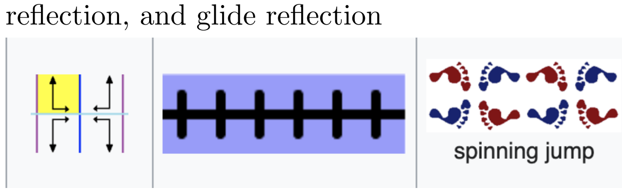
(d) TG: Translation and glide reflection



(e) TR: Translation and 180° rotation

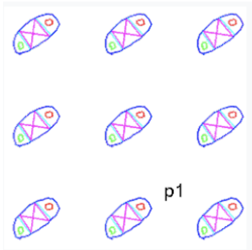


(f) TRVG: Translation, 180° rotation, vertical line reflection, and glide reflection



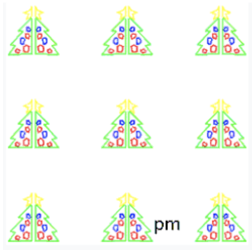
(g) TRHVG: translation, 180° rotation, horizontal line reflection, vertical line reflection, and glide reflection

Figure 7: Frieze Groups

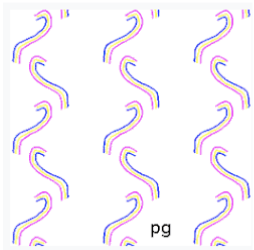


(a) p1: Translations only

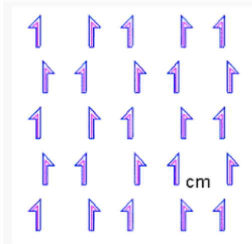
(b) p2: Contains only 180° rotations



(c) pm: Parallel reflection axes, no rotation.



(d) pg: Glide symmetries only with parallel axes, no rotations or reflections.



(e) cm: Reflections, at least one glide.

Figure 8: Some of the Wallpaper Groups