The Application of Group Theory to Music Theory

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Abstract

Mathematics has been often used to describe several aspects of music, from the acoustics of a vibrating string to algebraic music theory. In this paper, I examine the connection between group theory and music theory. In particular, I use cyclic groups to describe the twelve-tone scale of Western music as well as the circle of fifths. I first introduce a little bit of historical background to the mathematics of music. I then describe a few basic concepts of Western music theory, namely the twelve-tone equal temperament scale and the sequence of intervals known as the circle of fifths. Thirdly, I introduce definitions from group theory. Fourthly, I show how group theory and music relate. Finally, I reflect on my presentation on this topic, the feedback that I received from my peers, and potential areas of future research.

1 Introduction

1.1 Mathematics and Music in the Past

There is a close kinship between mathematics and music, one in which many technical aspects of music can be thoroughly and consistently described by mathematics. Many in the past have noted and studied this connection between the two. As early as the fourteenth to tenth century B.C., the Babylonians in ancient Mesopotamia recorded the numerical relationships of intervals on a string (see Fig. 1). In particular, they were aware of the octave, the interval of two notes in which one is double the frequency of the other. [1] The mathematical treatment of music continued in later cultures. In ancient Greece, Pythagoras studied the ratios between frequencies of pitches, e.g., that the interval of a perfect fifth corresponds to a 3:2 ratio in a tuning system known as *just intonation*. [2] Later, in universities in medieval Europe, music was studied as part of the *quadrivium*, the group of subjects that included arithmetic, geometry, astronomy, and music. In short, music and mathematics have long been "academic neighbors."

Furthermore, as mathematics has developed, it has given us more sophisticated tools to better describe different aspects of music theory. For instance, we now have a greater understanding of timbre—the sonic "character" or quality of an instrument,

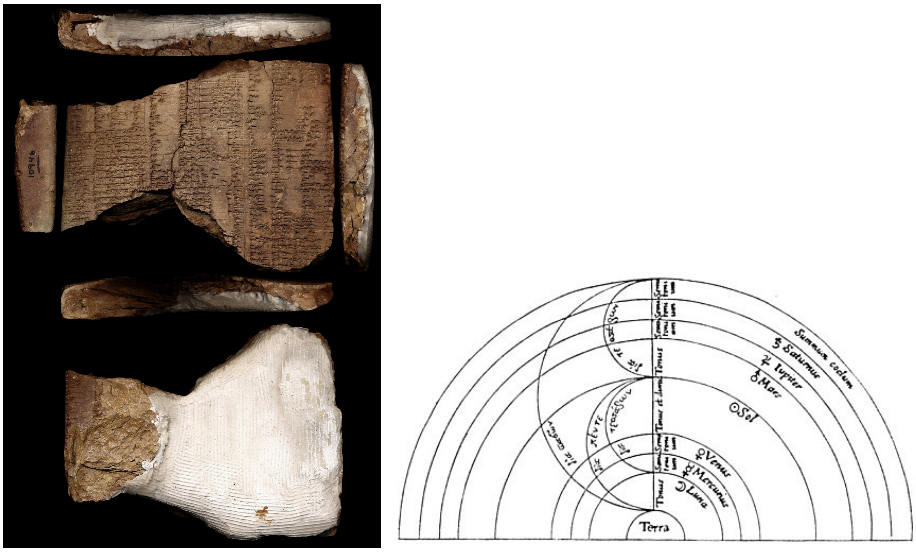

Fig. 1 Left: Middle Babylonian clay tablet from ca. 1400–1100 B.C. describing mathematical relationships of musical intervals. [3] Right: Diagram representing the "music of the spheres," the idea promulgated by the Pythagoreans that the planets make music, highlighting the link between mathematics, astronomy, and music as in the *quadrivium*.[4]

such as the warmth of a cello—via our better understanding of acoustics and the harmonic series, which is the sequence of frequencies of a pitch that includes a fundamental frequency and the frequencies simultaneously sounding above it known as overtones. To give another example of more modern mathematics describing music theory, one can relate musical chords—two or more notes sounding at once—to non-Euclidean geometry, as shown by Dimitri Tymoczko. [5] These are just a few examples of many, and I briefly note them only to demonstrate that the mathematical applications to music are wide-reaching, as they branch into a surprising variety of mathematical subjects like physics and modern geometry.

1.2 Group Theory, Music, and My Goals

Within mathematics' wide and historied applicability to music, I would like to focus on one particular application: the application of group theory to the musical scale. I would like to focus on group theory for two primary reasons. First, group theory is quite an intuitive tool to describe music theory. As we will see soon, it is simple and natural to represent the collection of pitches in a musical scale as integers, and from this simple numerical representation follows a good bit of introductory group theory, all of which is usually covered in a first undergraduate course in abstract algebra. Second, I would like to demonstrate the connection between group theory and music in order to convince some that group theory, though it is often categorized as a branch of "abstract algebra," is not as abstract or isolated from other subjects as some might believe. In particular, I wish to demonstrate the relevance of group theory to the concept of the circle of fifths, a foundational concept in modern Western harmonic practice. Indeed,

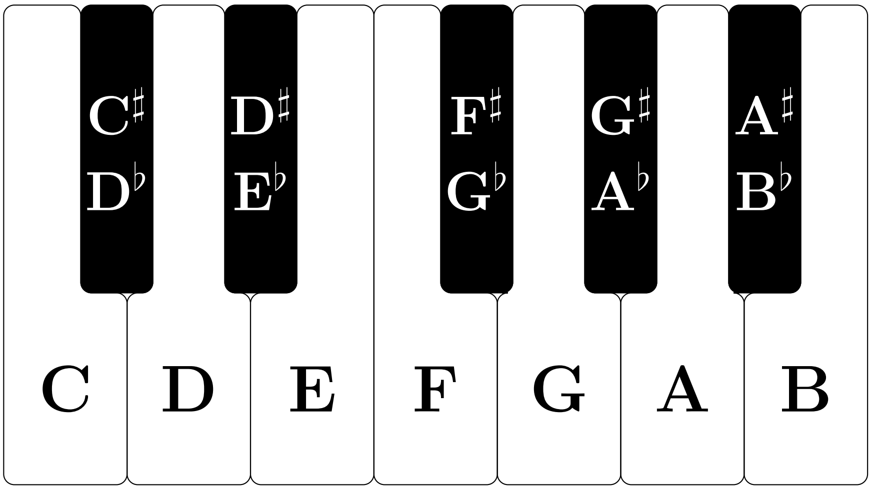

Fig. 2 The twelve-tone scale represented on a keyboard. All twelve notes in the scale are represented on this single octave; the next note above/to the right of B is another C, which has the frequency exactly double that of the C pictured on the left. (The black keys have two names, but the two notes on the key are **enharmonic**, i.e., sonically equivalent.)

just as the Mesopotamians and Greeks saw music and the mathematics of their day intertwined, today we can link our modern, "abstract" mathematics with music.

2 The Musical Scale and the Circle of Fifths

2.1 The Musical Scale

Let us first examine the musical scale most commonly used in modern Western music. This scale, consisting of twelve, equally spaced pitches, uses a tonal system known as *twelve-tone equal temperament*. The distance or interval between two adjacent pitches is known as a *semitone* or *half-step*, and the ratio between the frequencies of these adjacent pitches is $2^{\frac{1}{12}}$. The interval between twelve semitones is known as an *octave*, and the ratio between an octave is 2, i.e., a pitch exactly an octave above another pitch has double the frequency. We regard two pitches separated by an octave to be the same note, and they have essentially the same harmonic function. For instance, a C major chord (comprised of the notes C, E, G) is *essentially* the same (and is named the same) if it is played an octave higher or lower. Thus, the twelve pitches within the octave correspond to all the notes in our tuning system, and we may safely ignore all but a single octave for our discussion.

Twelve-tone equal temperament is far from a universal standard. For instance, the equally spaced intervals only roughly correspond to the aforementioned Greeks' system of just intonation. In fact, equal temperament arose only in the sixteenth century, and even within late Renaissance and Baroque European music, composers and theorists used a variety of twelve-tone tunings. [6] Additionally, there exist in the world other tuning systems that use not twelve pitches but five, seven, nine, or twenty-four pitches. I do not consider these systems in this paper, but the group theoretic principles may still apply to these alternatives. Furthermore, the fact of equal-temperedness is often a mathematical idealization, as many instruments are not tuned to have exactly the same distance between each semitone. [7] For the sake of this paper, however, the differences are negligible, and we only must assume that we have twelve pitches that are roughly equally spaced across the octave.

2.2 The Circle of Fifths

Within the twelve-tone system, there is an important harmonic concept known as the **circle of fifths**. An interval of seven semitones is known as a *perfect fifth*, which roughly has a 3:2 ratio between frequencies of pitches, roughly corresponding to the Greeks' just intonation. The simultaneous sounding of two pitches a perfect fifth apart is particularly consonant, i.e., pleasant-sounding. If one repeatedly ascends by perfect fifths, one plays each note in the twelve-tone scale until the whole pattern repeats. This pattern is useful for creating chord progressions as well as for **modulation**–that is, changing between keys or tonal centers. Take, for example, a piece written in the key of C major. C major consists of the notes C, D, E, F, G, A, B, C. Since G is a perfect fifth above C, the key of G major is harmonically similar to the key of C, differing only by one note. Indeed, the key of G major consists of the notes $G, A, B, C, D, E, F\sharp$, which is identical to C major except for $F\sharp$. In other words, pitches that are adjacent or close to each other on the circle of fifths are a sort of "harmonic neighbors," whereas pitches that are far from each other on the circle of fifths are harmonically disparate, sharing few common pitches. Because the circle of fifths contains all twelve pitches without repeats, a composer could theoretically modulate a piece twelve times until returning to the original key, only using "smooth" modulations a perfect fifth apart.

The circle of fifths has been widely employed by composers from the Baroque period in the seventeenth and eighteenth centuries such as Antonio Vivaldi and J.S. Bach as well as by modern jazz musicians, and it remains a fundamental characteristic of Western harmony. It is this concept that I ultimately aim to explain via group theory.

3 Some Basic Definitions from Group Theory

Allow me to briefly introduce some concepts from group theory that will aide us in the following discussion. Some understanding of set notation is requisite.

Definition 1. A **group** $(G, +)$, or, often denoted just by G, is a set equipped with an operation denoted by $+$ such that the following properties hold:

- $a + b \in G$ for all $a, b \in G$ (closure under operation)
- There exists $0 \in G$ such that $a + 0 = a$ for all $a \in G$ (where 0 is known as the **identity element**)
- For all $a \in G$, there exists $-a$ such that $a + (-a) = 0$ (where $-a$ is called the **inverse of** a). We commonly call adding the inverse **subtraction.**
- $a + (b + c) = (a + b) + c$ for all $a, b, c \in G$ (associativity)

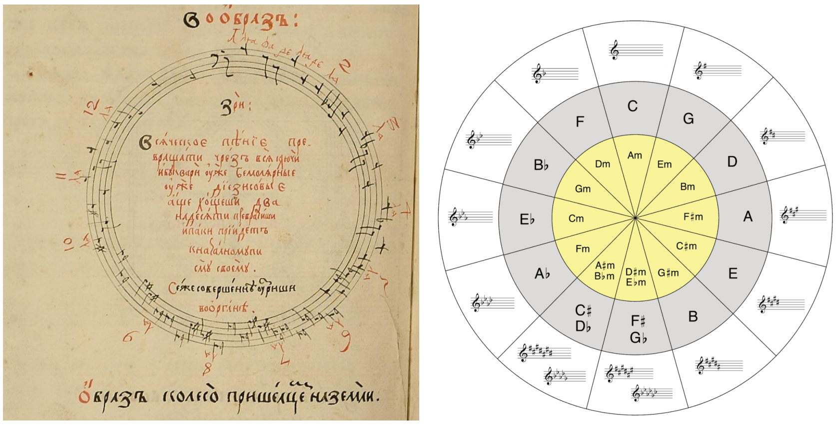

Fig. 3 Left: The earliest known circle of fifths diagram, drawn by music theorist Nikolai Diletskii's *Idea Grammatikii Musikiyskoy* from the late 1670s. [8] Right: A modern circle of fifths diagram; the middle gray ring shows the pitch name, the outer white ring shows the key signature corresponding to each pitch as the root of the key, and the inner yellow ring shows the relative minor chords corresponding to each pitch. For the sake of our discussion, we may ignore all but the middle ring. One can observe that, by moving clockwise around the circle, the pitches move up by perfect fifths through all twelve notes.[9]

Definition 2. A **subgroup** H of a group G is a subset of G that is also a group under the same operation.

Definition 3. Let G be a group, and $a \in G$. Then the **cyclic subgroup of** G **generated by** a is $\langle a \rangle := \{na : n \in \mathbb{Z}\}$. In other words, it is all integer multiples of a. I.e., it contains $a, a + a = 2a, a + a + a = 3a, \ldots$ as well as $(-a) + (-a) = -2a, (-a) + (-a) + (-a) = -3a, \ldots$.
If $G = \langle a \rangle$, then G is called a **cyclic group**.

This is all the group theory knowledge one needs to get a decent start on algebraic concepts in music theory. Please reference these definitions as needed in the following discussion.

4 Integrating Group Theory with the Musical Scale

4.1 Building a Group from the Scale

Now let us mathematize the twelve-tone scale. We begin by identifying each pitch $C, C\sharp/D\flat, D, \ldots, A\sharp/B\flat, B$ with the integers $0, \ldots 11$ in ascending order. Within this numeric system, how may we move between notes? We may consider moving up to a pitch to be the addition $(+)$ operation. Then, moving down is simply subtraction, or, more precisely, adding the inverse. For example, if we move up from C to G, then we are moving up 7 semitones. This corresponds to the statement $0 + 7 = 7$. Likewise, if we move down from C to G we are moving down 5 semitones. This corresponds to the statement $0 + (-5) = -5$, which, since the pitch number of G is 7, we have the fact that $7 = -5$ in our system.[1] This implies that $7 + 5 = (-5) + 5 = 0$, which is equivalent to saying that 7 is the (additive) inverse of 5 in our scale! We may repeat the same process for moving up or down to any pitch to observe that every movement up the scale has an inverse movement down the scale.

Indeed, we may simply regard the interval (or distance) between any two pitches to be exactly *the difference of their pitch numbers*. For example, the distance up from D to A is again 7 semitones, which is exactly the difference between their pitch numbers, $8-1$. Thus, we may also identify any interval with a pitch number x, so that $x = x-0$, i.e., the pitch number is equivalent to the interval from C up to the corresponding pitch. The names of these intervals is recorded in Table 1.

Table 1 Pitch Names, Pitch Numbers, and Interval Names

Name of Pitch	Pitch Number	Interval Above C
C	0	Unison
$C\sharp/D\flat$	1	Semitone / Minor Second
D	2	Major Second
$D\sharp/E\flat$	3	Minor Third
E	4	Major Third
F	5	Perfect Fourth
$F\sharp/G\flat$	6	Diminished Fifth / Tritone
G	7	Perfect Fifth
$G\sharp/A\flat$	8	Minor Sixth
A	9	Major Sixth
$A\sharp/B\flat$	10	Minor Seventh
B	11	Major Seventh

This scale represented by pitch numbers seems to resemble closely a group, but let us first (informally) verify that the properties of a group hold. We have already seen that there exists an inverse element for each pitch number in the scale. We have seen that the scale contains C, which is identified with 0, which has the property that $0 + x = x$ for all pitch numbers x. We also ought to show that the scale is "closed" under movement between pitches and that this movement is associative.

First, there are no other pitches than the 12 listed in this tuning system. In other words, moving between any two pitches will never "escape" the scale. But consider starting at C and moving up two perfect fifths. This corresponds to the operations $0+7+7 = 14$. But 14 is not a pitch number of our scale! However, because a perfect fifth above G is D (which is regarded as the same pitch as the D that is a perfect fourth *below* G), we have that $14 = 2$.[2] In any sequence of intervals, regardless of whether we move to a different *octave*, we never move outside the scale, since the pitches of each octave are regarded as equivalent.

Lastly, we may see that moving between pitches is associative. For example, consider this sequence of three movements: Starting on C, move up a semitone, then

[1] Please note that I am being loose with notation. I, of course, do not mean that $7 = -5$ as integers. Rather, I mean that, as *equivalence classes*, we have $[7] = [-5]$, or rather that $7 \equiv -5 \mod 12$. The notion of equivalence classes helps to formalize the idea that different octaves are "equivalent," but it does not change the basic ideas presented. I hence omitted it from the main discussion.

[2] See previous footnote.

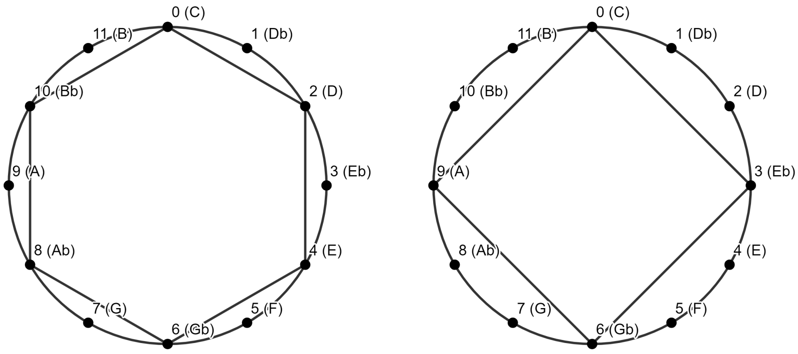

Fig. 4 Left: Circle diagram of $\langle 2 \rangle$. Right: Circle diagram of $\langle 3 \rangle$ (right). These are proper subgroups of S.

up a major third, then down a major second. We can combine the upward semitone and upward major third first to obtain an upward perfect fourth. Alternatively, we can combine the upward major third and downward major second first to obtain an upward major second. Either way, this sequence still ends on $E\flat$. Or, in mathematical notation, $(1+4)+(-2) = 1+(4+(-2)) = 3$. We may generalize this to any sequence of intervals to show that movement between pitches is associative.

Therefore, the twelve-tone scale equipped with the operation "moving up" an interval upholds is, in fact, a group. Those acquainted with modular arithmetic may note that this group is *exactly* the integers modulo 12, denoted by $\mathbb{Z}_{12}$, under addition! From now on, we may call this group the **twelve-tone scale group**, or simply S.

4.2 Cyclic Subgroups of the Twelve-tone Scale Group

Now that we know that S is a group, let us consider its subgroups–more specifically, its cyclic subgroups. Consider the set $\langle 2 \rangle = \{2n : n \in \mathbb{Z}\} = \{0, 2, 4, 6, 8, 10\}$. This is a proper subset of S, and it may be easily verified that it is also a group. In other words, $\langle 2 \rangle$ is a proper subgroup of S. On the scale, this set corresponds to the set $\{C, D, E, G\flat, A\flat, B\flat\}$. This is what is known in music theory as a **whole tone scale**, a scale of six pitches that are each separated by a major second (or whole tone).

We may consider other such cyclic subgroups of S. Observe that $\langle 3 \rangle = \{0, 3, 6, 9\}$ is likewise a proper subgroup of S, which corresponds to the set $\{C, E\flat, G\flat, A\}$ which is known in music theory as a **diminished seventh chord**, whose pitches are all separated by minor thirds.

But now consider the subgroup $\langle 7 \rangle$. This corresponds precisely to the circle of fifths, which results in the set $\{0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5\} = \{0, 1, \ldots, 11\} = S$. Here we see that $\langle 7 \rangle$ is not a *proper* subgroup of S but is S itself–in other words, S is a cyclic group.

We may use every other element of S as a generator of a cyclic subgroup (see Table 2). This naturally leads us to a question: Why do some elements of S generate

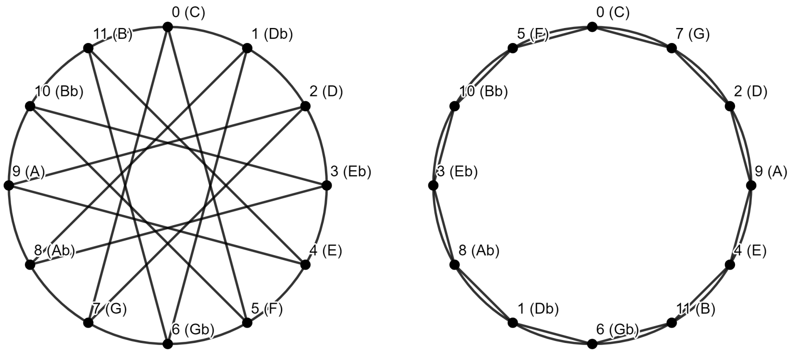

Fig. 5 Left: Circle diagram of $\langle 7\rangle$. Right: When the nodes on the circle are rearranged such that nodes connected by a line segment are adjacent, we obtain a Circle of Fifths diagram.

Table 2 Cyclic Subgroups of S and Corresponding Sets of Pitches

Generator	Set of Pitches	Common Name in Music Theory
0[3]	$\{C\}$	—
1	$\{C, D\flat, E, E\flat, F, G, A\flat, A, B\flat, B\}$	Twelve-tone (chromatic) scale
2	$\{C, D, E, G\flat, A\flat, B\flat\}$	Whole-tone scale
3	$\{C, E\flat, G\flat, A\}$	Diminished seventh chord
4	$\{C, E, A\flat\}$	Augmented chord
5	$\{C, D\flat, E, E\flat, F, G, A\flat, A, B\flat, B\}$	Twelve-tone (chromatic) scale
6	$\{C, G\flat\}$	Tritone
7	$\{C, D\flat, E, E\flat, F, G, A\flat, A, B\flat, B\}$	Twelve-tone (chromatic) scale
8	$\{C, E, A\flat\}$	Augmented chord
9	$\{C, E\flat, G\flat, A\}$	Diminished seventh chord
10	$\{C, D, E, G\flat, A\flat, B\flat\}$	Whole-tone scale
11	$\{C, D\flat, E, E\flat, F, G, A\flat, A, B\flat, B\}$	Twelve-tone (chromatic) scale

a proper subgroup of S where another generates S itself?

To answer this question, we turn to a theorem that makes a statement not just about S, but all cyclic groups:

Theorem 1. Let $G = \langle a\rangle$ be a cyclic group with n elements. Furthermore, let $k \in \mathbb{Z}$. Then $G = \langle ka\rangle$ if and only if $\gcd(k, n) = 1$, i.e., k and n are relatively prime.

Let us see this theorem in action, as applied to S. We can see by Table 2 that $S = \langle 1\rangle$. Then $S = \langle k1\rangle = \langle k\rangle$ if and only if $\gcd(12, k) = 1$. In other words, if $k = 1, 5, 7, 11$, then $S = \langle k\rangle$; otherwise, if $k = 0, 2, 3, 4, 6, 8, 9, 10$, then $S \neq \langle k\rangle$.

Without this result of group theory, all of modern Western music would be bereft of the circle of fifths and the harmonic techniques it entails.

[3] I.e., it generates the "trivial subgroup" $\{0\}$ consisting only of the identity.

5 Reflection

5.1 Personal Reflection

Ever since I took my first undergraduate course in algebra, I have attempted to see many other subjects–mathematical or otherwise–from an algebraic lens. Group theory, with its formalization of basic operations and sets, allowed me to see mathematics from a new, more basic, more fundamental, perspective. For me, group theory was, as a first foray into abstract algebra, a formative subject, teaching me how mathematicians have dug into the foundations of mathematics have built it up very carefully from simple axioms. By learning about these strange mathematical objects like groups, rings, modules, etc., I found that there is a creative spirit in doing mathematics, one that compels me to study and to do more mathematics. Because of my fascination with algebra, I naturally tend to look for algebraic objects out and about in the wild. For instance, now I think of clocks and timekeeping from the perspective of groups. Meanwhile, as a musician, I also was studying music theory, and for whatever reason, it did not occur to me that group theory might crop up in music. Indeed, when someone pointed out to me this phenomenon of the cyclic subgroups of the twelve-tone scale, I was taken aback. How is it, that the two things I enjoy studying most deeply, are actually interconnected? It was thus a pleasure to research more about algebra in music theory for this project, and I was happy to see that many others studying the two fields have made similar observations.

I also contemplated the broader connection between mathematics and music. Some musicians resist any attempt to mathematize music, as if doing so reduces art to a cold, unfeeling science. I am sympathetic to this view, as I do not believe one can embrace the essence of music through the study of music theory, however mathematical it may be. Rather, in learning how group theory relates deeply to music theory, I see that, on the converse, *mathematics* is more musical–that is, mathematics has its own inner logic and beauty that is charming in much the same way that music appeals to our aesthetic sense. In this way, I find mathematics and music similarly affecting.

Additionally, I enjoyed sharing my interest in the topic with others in the course during my presentation. Regarding the presentation specifically, I learned that I do not mind getting up in front of a class to lecture about mathematics. However, I also noticed myself often assuming that the audience knew what I meant even when I did not explain a concept clearly, which helps let me know that I ought to practice mathematical communication more so that I can effectively transmit my fascination with the subject to others.

5.2 Peer Reflection

I am happy with the feedback that I received from the class. People generally seemed to enjoy the presentation, and many were surprised to see how the topics in their past abstract algebra class related to music. Those without any musical background, however, said that they struggled to understand my explanations of music theory. I believe that I covered the material too quickly in my presentation, and so in this paper I took more time to thoroughly explain the scale and the circle of fifths.

5.3 Areas of Further Research

Because the mathematics of music theory is so deep, there is still much more to say about group theory, or more broadly algebra, as it relates to music. I mentioned earlier that there are other tuning systems beside the twelve-tone equal temperament scale that is common in Western music, such as the pentatonic, heptatonic, and 24-tone scales. One can consider these to be $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_{24}$, respectively, and one may examine the cyclic subgroups of each.

During class, I also suggested the direct product of these groups, such as $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$. This product may be interpreted as an ensemble of two instruments playing at once. This can be generalized to an ensemble of n instruments, which may produce interesting algebraic results involving n-dimensional vectors.

Given the breadth of both algebra and music theory, there are many more avenues that one could take to explore this connection more deeply.

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