Journey of the Continuum Hypothesis

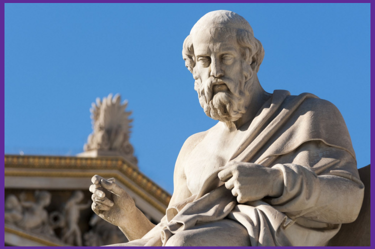
Kyle Gomez

Background

- In talking about the Continuum Hypothesis and the set theory surrounding it, there are many places to start
  - Cantor's establishment of set theory and proposal of the problem would probably be a good place to start
  - I started slightly earlier than this, in an effort to show Cantor's career building to this point
- Naive set theory used implicitly in math since at least Aristotle (384–322 BCE)
- For many years, people mostly worked on higher subjects in math, thinking that there wasn't much of interest to be found in the basic principles they had been assuming for forever
- "Formalization" of math took place over many years and spread through different subjects at different rates
- Gradually became a push toward *axiomatizing* math
  - Enumerating the basic principles, and building everything else formally from that point

Cantor's early career

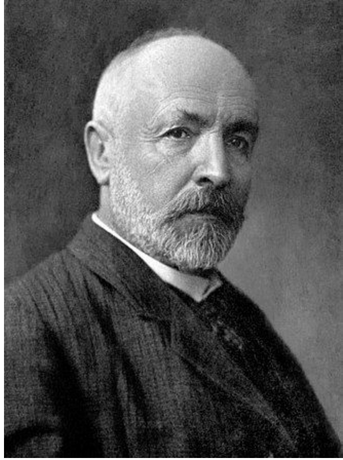
How did the "founder of set theory" stumble into the topic?



"The beginning is the most important part of the work."

–Plato, *The Republic*

Georg Cantor (1845–1918)



- German-Russian mathematician at the end of the 19th century
- One of the first people to begin enumeration of set theory
- At the beginning of his career, he worked with number theory for a short time
- Edward Heine (1821–1881) was a fellow mathematician who convinced Cantor to switch from working with number theory into functional analysis
    - This would lead almost directly into his work with set theory

Cantor’s early analysis work

- Heine specifically asked Cantor to work on the question of whether a function represented by a trigonometric series was unique
  - Heine had already proved this for a specific series
- Dirichlet, Lipschitz, and Riemann had all failed at completing this endeavor
  - Foreshadowing
- He focused on representing 0 with trigonometric series and soon found that it was equivalent to the coefficients being 0, which allowed him to make the general statement that a trigonometric series is unique for a function it represents
- He was able to solve this problem within the same year Heine had published his more specific solution (1870)

Cantor's uniqueness developments

- Cantor's original uniqueness theorem required that the trigonometric series that represents f(x) was convergent for every value of x
- He generalized it further, allowing the representations to be divergent at some values of x ("exceptional values", leading to the idea of *finite exceptional sets*)
- Cantor thus wanted to generalize it further to *infinite exceptional sets*, but required a more rigorous theory of real numbers
- He built irrationals based off of the limits of convergent sequences of rational numbers, so that any real number could be expressed from rationals, in order to build a connection to the real line

Derived sets

- To the same ends, Cantor took a “set of points” (not fully defined yet) P and considered the set of limit points of P, called P’
  - This was the *first derived point set* of P
  - If P was the set of rational numbers, P’ would be the real numbers, as shown before by treating irrationals as limits of rational sequences
- Continue this process to find $P^{(v)}$ for some natural number v
- It became interesting when $P^{(v)}$ would have only a finite number of points, and thus $P^{(v+1)}$ didn’t exist
  - Such sets that had a finite vth kind were part of the derived sets of the first species
- He was able to use such derived sets of the first species to show that the uniqueness theorem worked for these infinite exceptional sets

Second species of derived sets

- The next question was what happened when $P^{(v+1)}$ would still exist for any *finite* number v
- The second species of derived sets were the ones where v was infinite, and thus could be built off even further
- This whole example highlights how ready Cantor was to jump forward by generalizing further toward infinite processes in any way imaginable, something that would characterize his greatest accomplishments
  - In fact, Cantor would use this idea of derived sets constantly, and thought about his later infinite ideas as extensions of this same idea
- Dedekind criticized this whole process, since while deriving from the rationals gave the real numbers, deriving from the real numbers just gave the real numbers over and over
- We're lucky Cantor didn't listen to this, since he wouldn't have found his most famous results if he hadn't continued down this road of abstracting to infinity

Cantor's analysis work as a stepping stone

- We saw earlier the deriving process to $P^{(\infty)}$, and in this context Cantor had called ∞ the first number beyond all finite numbers, but he hadn't really built this notion
  - He had even written $P^{(\infty')}$ as $P^{(\infty+1)}$, a subtle establishment of later ideas of transfinite ordinal arithmetic
- His work in analysis (in particular with irrationals) also led him to ask **whether there was a correspondence between the set of integers and the set of real numbers** (in an 1873 letter to Dedekind)
- "On a Property of the Collection of all Real Algebraic Numbers" (1874) he published his first proof that the set of real numbers was nondenumerable
  - Interestingly to us, this was separate from his diagonalization argument we associate now with the statement

Cantor's first uncountability proof

- By contradiction, assume the real numbers are countable and list w_1, w_2, w_3 …
- Let $a<b$ be real numbers. Then let a', b' be the first two numbers within the countable set of real numbers within the open interval (a,b)
- Similarly to his derived sets from before, continue this process to an arbitrary nested open interval $(a^{(v)},b^{(v)})$ for some v
- If this is a finite process: that meant there was only one more of the w_1, w_2, w_3, … in the interval, so you can still find a real number that is in the interval and not enumerated
- If the process is infinite: a^∞, b^∞ must be bounded by (a,b).
    - If $a^\infty<b^\infty$, just as above there is a real number in the interval not enumerated
    - If $a^\infty=b^\infty$, then this element was the one that couldn't be enumerated in the original list, since all enumerables have already been excluded as the endpoints of earlier open intervals

Cantor uncountability consequences

- This process is still used for actually numerable sets to find digits of transcendental numbers
- Cantor actually used this for any real interval, to show that any interval was uncountable, not just the whole real line
- Cantor developed and redid this proof multiple ways, including one with topological density, and the more famous diagonalization method
    - We don’t need to go through all these
- This led Cantor further into studying the infinite, and building a strong base of set theory to allow for this

Cantor's Set Theory

What the hell are transfinite numbers??

"Some infinities are bigger than other infinities"

–John Green, *The Fault in Our Stars*

–Probably Cantor too at some point when he discovered this

Beginnings of set theory

- In order to build to his idea of "different infinities," Cantor started from scratch essentially
- First sentence of a later text on set theory (the *Beiträge*): "By a 'set' we mean any collection M into a whole of definite, distinct objects m (which are called the 'elements' of M) of our perception or of our thought."
- This is still what would be considered naive set theory, but this was one of the first times someone actually tried to enumerate what a set was, and use that as the first step in a process

Ordinals vs cardinals

- Natural numbers implicitly play two simultaneous roles:
- Using them to indicate size/cardinality of a set (cardinals)
- Using them to indicate order (ordinals)
- With finite numbers, these don’t have a real difference, but Cantor discovered they absolutely do when it comes to infinite numbers, so we must make the distinction now

Transfinite ordinals

- Consider the set of natural numbers. Notice how it consists of repeated addition of units
  - Cantor calls this the first principle of generation
- Now consider the first number after this entire set. This is the first transfinite number, □
  - It's possible to consider this number as a limit of the natural numbers that they can never reach
- Now consider the first number after this, □ + 1
- This is where the distinction between ordinals and cardinals becomes important:
  - For ordinals, □ + 1 is distinct from □ since it is generated after it
  - If considered as cardinals, □ + 1 would not be distinct from □ since they have the same cardinality (in fact, this idea of where ordinals become distinct as cardinals is essentially the focus of this presentation)

Transfinite ordinals

- You can continue farther to get □ + 2, □ + 3, … , □ + v, … , □ + □ = □ · 2
- Transfinite ordinal arithmetic is strange compared to integers, since it's not commutative
- This idea of taking a succession of ordinals which has no end and "jumping ahead" to the first ordinal after all of those is what Cantor called the *second principle of generation*
- You can continue these principles to get many different transfinite numbers
    - One could even say, an infinite amount of different transfinite numbers

Classes of numbers

- Using his idea of transfinite numbers, he called the class of all finite whole numbers the first number class (I)
- He then defines the second number class (II) as the collection of all transfinite ordinals formed by the two principles of generation, with the stipulation that each element of (II) has predecessors forming a countably infinite set
- Similarly to his uncountability proof, Cantor showed that the cardinality of (I) was strictly less than that of (II)
- He was even able to show that (II) had the very first cardinality higher than (I)
- Since (I) corresponded to the cardinality of the natural numbers, **Cantor assumed based on his earlier work that (II) corresponded to the cardinality of the real numbers, and thus that the cardinality of the real numbers was just after the cardinality of the natural numbers**

The Continuum Hypothesis

- We’ll adjust our statement to be more concise in a minute based on study of transfinite cardinals, but this statement just now was in fact the continuum hypothesis
- Cantor specifically stated “it will be concluded that the linear continuum has the power of the second number class (II)”
  - “Power” was what he called the cardinality of sets at the time (I thought this would be confusing to use regularly)
- This was the idea that the cardinality of the real numbers was the next highest cardinality after that of the natural numbers, that you can’t find a set of a size strictly between that of the natural numbers and real numbers
- Cantor would work his whole life trying to prove this, and **never could**
- He developed further a lot more about set theory and transfinite numbers, largely in pursuit of proving this statement

Transfinite cardinals

- Remember cardinals correspond to the order/cardinality of a set
- Two sets have the same cardinality if there is a one-to-one correspondence between them
- For finite sets, our cardinal numbers are still the natural numbers that are also finite ordinal numbers
- Define the cardinality of the set of all the finite cardinal numbers to be $\aleph_0$
  - This is the first transfinite cardinal number, called aleph null
- So the cardinality of the natural numbers is equal to $|\square| = |\square+1| = \ldots = |\square \cdot 2| = \ldots = \aleph_0$
- Cantor showed that while many operations of transfinite cardinal numbers simply produced the same cardinal, it was always possible to find a higher one by raising it to the power of 2
  - $2^{\aleph_0} > \aleph_0$

Transfinite cardinals

- Now consider the class of all countable transfinite *ordinal* numbers (recall this is also the second number class (II))
- Cantor shows that the cardinality of this class must be greater than $\aleph_0$, and in fact that it is the very next transfinite cardinal number, $\aleph_1$
- He also shows that the cardinality of the continuum is $2^{\aleph_0}$
- This is another way of seeing that the set of real numbers is larger than the set of natural numbers, since $\aleph_0 < 2^{\aleph_0}$

Restating the Continuum Hypothesis

- Now we know that $\aleph_1$ is the cardinality just after $\aleph_0$, which is the cardinality of the natural numbers
- So Cantor's idea of the size of the continuum corresponding to the very next cardinality after the size of the natural numbers can be written succinctly as:
  - $2^{\aleph_0} = \aleph_1$
- Cantor never proved or disproved this in his lifetime, and so this is where we leave him

Development after Cantor

Popularization of his work and discovery of paradoxes in set theory

“This sentence is false.”

“Um, true. I’ll go with true. There, that was easy. To be honest, I might have heard that one before.”

“It’s a paradox! There is no answer.”

–Portal 2

Hilbert's first problem

- David Hilbert (1862–1943) was known as a defender of Cantor's set theory and transfinite numbers
- Hilbert made a speech at the 1900 conference of the International Congress of Mathematicians about multiple unsolved problems in math which could have important implications
- He later published a list of these problems and more (23 in total) that he wanted solutions for, known as Hilbert's Problems
- The first of both of these lists was the continuum hypothesis, indicating that it was very important to Hilbert and the wider math community
- Hilbert wanted formal axiomatic systems but also wanted there to be a proof or a counterexample to each problem, under the belief that every true statement has a proof
    - As have seen and soon will see, this would not be possible for more than one of his problems

Burali-Forti's Paradox

- Consider the succession of all ordinal numbers
- Then there is an ordinal number greater than all the ordinal numbers
- But then this collection must also include the ordinal number strictly greater than all the ordinals in the collection
  - Contradiction

Russell’s Paradox

- Let R be the set of all sets that are not members of themselves
- Then R isn’t a member of itself, but then it must be a member of itself, but then it can’t, and so on
- Yet another paradox inherent in Cantor’s set theory

Resolutions

- There was some back and forth between mathematicians about the implications of these paradoxes, and how much of Cantor's set theory they were able to undo
- There were many sides and positions, some thinking that certain properties were no longer consistent, some thinking that there were some implicit assumptions that shouldn't have been made
- Ultimately, multiple people saw the solution to this to be to fully axiomatize set theory, and make it so that these inconsistencies couldn't exist in the first place
- There were multiple attempts at this, but Ernst Zermelo would make the most impact

Axiomatic Set Theory

Zermelo and Fraenkel develop their set of axioms about sets

"Smokey, this is not Nam, this is bowling, there are rules."

-The Big Lebowski

Zermelo-Fraenkel Set Theory

- Zermelo listed 7 axioms that could define an entire base of set theory
- Fraenkel later noticed that these axioms couldn’t prove the existence of certain sets and cardinal numbers that mathematicians would want to work with, so he amended an axiom and added another
- ZF is Zermelo-Fraenkel Set Theory
- It would later be shown that the Axiom of Choice is independent of ZF, so ZF with the Axiom of Choice is called ZFC
- ZFC is the standard model of set theory used today

Basis of ZFC

- Assume nothing except these logical symbols:

  $\sim$: not
  $\wedge$: and
  $\vee$: or
  $\Rightarrow$: implies
  $\Leftrightarrow$: if and only if
  $\forall$: for all
  $\exists$: there exists

- And also any variable is a set. Don't ask what a set is yet. It will be implicitly defined by the axioms, along with multiple other concepts
- Then ZFC is these 9 axioms usually written completely logically from these
  - Technically there are a couple different constructions equivalent to ZFC (of course) so I'll be following Fraenkel's amended list, with the axiom of choice added (this is what Cohen used in his book)
- We're going to just gloss over them without the full logic

1 Axiom of extensionality

Two sets are equal (are the same set) if they have the same elements

This also introduces the relation of membership, but the rest of the axioms work with this one to define it

2 Axiom of the null set

There is an empty set defined by saying that there are no elements in the set

3 Axiom of Unordered Pairs

If there are two sets, we can form another set out of the two of them in a set

4 Axiom of union

The union of two sets exists

5 Axiom of Infinity

If we have an ordinal, its successor will be the union of it and the set containing it

Then there is a set that contains the empty set, and if it contains an ordinal, it also contains its successor

This is what induction is based off of

6 Axiom of Replacement

If a statement describes a set uniquely as a function of a given set, then the image of any set under this function is itself a set

This is very carefully constructed to avoid allowing sets to be defined by ridiculous statements, and therefore preventing common paradoxes

7 Axiom of the Power Set

For any set, the set of all of its subsets exists

8 Axiom of Regularity

Every nonempty set has a minimal element with respect to membership

Also a set can’t be a member of itself

This often exists just to block paradoxes of self-reference

9 Axiom of Choice

For any set of nonempty sets, there is a choice function that maps each member set to an element of that set

There are many equivalent statements to this

“Given any family of nonempty sets, their Cartesian product is a nonempty set”

It’s since been discovered that this is a very important axiom to many areas of math, and also that it’s independent of the other axioms

Effects of ZFC

- It’s from these axioms that much of modern math is based
- ZFC is the standard set theory, and so most math is built off of these founding principles
- The axiom of infinity makes it so that we can’t technically say that this is a finite collection of axioms
- So far we haven’t found contradictions within this framework, but of course we know that we can’t prove the consistency of ZFC within ZFC
  - This result is thanks to Gödel
    - Foreshadowing

Gödel and Cohen's Results

Cantor and Hilbert's nightmare

"Deep in the human unconscious is a pervasive need for a logical universe that makes sense. But the real universe is always one step beyond logic."

–Frank Herbert, *Dune*

Gödel

- Kurt Gödel (1906–1978) was a German mathematician
- He greatly developed logic in math and the proof theory underlying all of math
- As we know, he showed that the consistency of a system cannot be proven within the system
- In 1937, Gödel also specifically proved that both the continuum hypothesis and the axiom of choice were consistent with ZF
- So from him, we're able to say both of these cannot be disproven from ZF

Cohen

- Paul J. Cohen (1934–2007) was an American mathematician
- Cohen developed a technique called forcing
  - This involves constructing an expanded universe from an old one with a new generic object, but specifically forcing it to have certain properties
- In 1963, he used it to show that not CH is also consistent with ZF(C)

Independence of the continuum hypothesis

- Together, since both CH and not CH are consistent with ZFC, we can conclude that **CH is independent of ZFC, or undecidable, that it cannot be proven one way or another within the system**
- Cantor wanted a yes, and Hilbert wanted a clear yes or no, so they probably both would have been disappointed by this result
- This proof of the independence of the continuum hypothesis from ZFC is considered very difficult, especially the proof by Cohen
- I would have given an overview of it, but it’s so technical and difficult that there both wouldn’t be enough time for the presentation, and I wouldn’t have had time to actually understand it

How did we get here? What did we learn?

- Think back to the original ideas: Cantor showed early on that the continuum is strictly larger than the integers
- He developed transfinite ordinals from this
- He had an idea that the continuum was the next largest level of infinity above the integers
- He developed transfinite cardinals and further fleshed out his basic set theory in pursuit of this question
- Hilbert brought multiple unsolved problems to the forefront of mathematicians' minds
- Zermelo developed axiomatic set theory based on the paradoxes found in naive set theory
- The axiom of choice was enumerated along the way
- Gödel developed incompleteness
- Cohen finished the job by showing the job can't be finished

Conclusions

- We now know that when using ZFC, we can simply choose whether we assume CH is true or not
- Many different theories and ideas in different branches of math were developed on the road to this solving this problem
- In particular, the formalization and axiomatization of math was developed along the way
- Unsolved problems are a fuel to mathematicians that feeds their fire and lights the flames revealing many other areas of math along the way
- There are still so many unsolved problems, and I hope we keep making up even more math to try to get at answers

References

- *Georg Cantor: His Mathematics and Philosophy of the Infinite*, Joseph Warren Dauben, 1979.
- *Set Theory and the Continuum Hypothesis*, Paul J. Cohen, 1966.
- "Independence of the Continuum Hypothesis: an Intuitive Introduction," Serafim Batzoglu, 2002.
- *Set Theory and the Continuum Problem*, Raymond M. Smullyan and Melvin Fitting, 1996.