Election Theory

Daniel Hopkins

1 Introduction

Election theory is a mathematical subject area that studies different electoral processes and methods. We will survey different categories of voting methods, look at various types of voting methods of each category, and discuss certain criteria that mathematicians, economists, and political theorists have decided is an important enough criterion to be considered an axiom of a democratic election.

2 Basic Terminology

Systems of voting in which voters are asked to state an ordinal preference between numerous candidates are called *preferential voting methods,* those systems that do not are called *approval voting methods*.

3 Voting Methods

The most primitive game that game theory describes are known as total-conflict games. These games provide no opportunity for cooperation because each player's objectives are completely antithetical to each other. That is, for one player to win, another player must lose.

3.1 The Plurality Method

In the *plurality method*, each voter selects one candidate on the ballot. The winner is the candidate with the most votes. Note that the winner does not need to have a majority of the votes.

For example, in a three-candidate election with 50 voters, candidate A gets 12 votes, candidate B gets 20 votes, and candidate C gets 18 votes. Thus, candidate B is the winner by plurality method, even though he does not have a majority of the votes.

3.2 Vote-for-Two Voting

Another simple voting method in elections where there are more than two candidates is *vote-for-two voting*. Each voter must vote for two different candidates and the candidate with the most votes wins. The idea is that this method should elect a candidate that is acceptable to most people.
For example, consider the case of the 1992 US presidential election:

Candidate	**36 Voters**	**8 Voters**	**30 Voters**	**9 Voters**	**7 Voters**	**13 Voters**
Clinton	✔	✔		✔	✔	
Bush		✔	✔	✔		✔
Perot	✔		✔		✔	✔

So each candidate's total votes are the sum of their rows, so Clinton earns 36+8+9+7=60 votes, Bush earns 8+30+9+13=48 votes, and Perot earns 36+30+7+13=86 votes, making Perot the winner. An unusual result as, we will find, as Perot was the third-party candidate who was almost no one’s top choice.

3.3 Preference Rankings

An important concept moving forward to discuss more complicated preferential voting systems is the idea of a ranked ballot, wherein a voter would rank all candidates from most favorable to least favorable. Again, our example is the 1992 presidential election.

Candidate	**36 Voters**	**8 Voters**	**30 Voters**	**9 Voters**	**7 Voters**	**13 Voters**
Clinton	1	1	3	2	2	3
Bush	3	2	1	1	3	2
Perot	2	3	2	3	1	1

When we look at this ticket using only simple plurality, we find that Clinton earns 36+8=44 votes, Bush earns 30+9=39 votes, and Perot earns 7+13=20 votes. Thus Clinton is the winner of this preferential ballot when we use simple plurality only.

3.4 Plurality with (Instant) Runoff

But preferential voting is supposed to reflect the preferences of the most voters it can, so we look to one method of using the information of the entire ballot, plurality with instant runoff. Before we define this voting method we must first make two assumptions about preferences:

1. If a voter ranked one candidate above another, then the voter would choose the higher-ranked candidate in a head-to-head election.
2. The order of preference is not changed if one or more of the candidates is eliminated, as in a runoff.

From these assumptions of voter preferences, we define a system where we assume that preference ballots have been used, and that no candidate has received a majority of the first-place votes. That is because if a candidate has one, then she would automatically become the winner of the election. Then, using the plurality assumptions above, the candidate with the least first place votes is removed from the election and the process is repeated until a candidate has a majority.

Candidate	**36 Voters**	**8 Voters**	**30 Voters**	**9 Voters**	**7 Voters**	**13 Voters**
Clinton	1	1	3	2	2	3
Bush	3	2	1	1	3	2
Perot	2	3	2	3	1	1

First, we compare first place votes amongst candidates, where Clinton earns 44, Bush 39, and Perot 20, so Perot's first place votes are then redistributed among the other candidates according to the preferences of the voters. As 7 of Perot's voters prefer Clinton over Bush, Clinton now has 44+7=51 voters, and as 13 of Perot's voters prefer Bush over Clinton, Bush now has 39+13=52 voters, so Bush would win in this election format.

3.5 The Borda Method

Even with instant runoff, the plurality methods fail to capture candidates you like, and candidates you hate. The Borda Method requires that a voter rank the N candidates, where first place is assigned 1, second place gets 2, all the way up to N points for a last-place vote. The candidate with the smallest point total is the Borda winner of the election.

Candidate	**36 Voters**	**8 Voters**	**30 Voters**	**9 Voters**	**7 Voters**	**13 Voters**
Clinton	1	1	3	2	2	3
Bush	3	2	1	1	3	2
Perot	2	3	2	3	1	1

So we get BC(Clinton) = 44 + 2(16) + 3(43) = 205, BC(Bush) = 39 + 2(21) + 3(43) = 210, and BC(Perot) = 20 + 2(66) + 3(17) = 203. So, Perot wins the election using the Borda method.

3.6 Condorcet Winner

Another method of determining the winner of an election when we know the preference rankings of each voter involves pitting each candidate against every other candidate in a series of head-to-head comparisons.

A candidate who is the winner of a head-to-head comparison with every other candidate is called a *Condorcet winner*. A candidate who is the loser of a head-to-head comparison with every other candidate is called a *Condorcet loser*. A given election may or may not have a Condorcet winner and/or loser. To see who wins in a head-to-head comparison between two candidates, ignore all other rows and compare the rank of the two candidates. Our preference assumptions ensure that a candidate with a higher rank is preferred to the other.

Candidate	36 Voters	8 Voters	30 Voters	9 Voters	7 Voters	13 Voters
Clinton	1	1	3	2	2	3
Bush	3	2	1	1	3	2
Perot	2	3	2	3	1	1

Clinton v. Bush: Clinton earns 36+8+7=51 votes, and Bush earns 30+9+12=52. Clinton v. Perot: Clinton earns 36+8+9=53 votes, and Perot has 30+7+13=50 votes. Bush v. Perot: Bush has 8+30+9 = 47 votes, and Perot has 36+7+13=56 votes. So there is **not** a Condorcet winner (or loser). The biggest issue with his voting method is that it often fails to even produce a winner at all.

3.7 Method of Pairwise Comparisons

In the last example, there was not a Condorcet winner of the election. Worse, the probability that there will be a Condorcet winner goes down significantly as the number of candidates increases. But, there still is a method, sometimes referred to as *Copeland's method*, is used to find the most victorious head-to-head competitor.

In each head-to head comparison, the winner is assigned 1 point, the loser 0 points, and in the case of a tie, each candidate is assigned 1/2 point. The overall winner of the election is the candidate with the most points after all head-to-head comparisons have taken place. Note that if there is a Condorcet winner, they will naturally win with Copeland's method.

For example, a new team is joining the NFL, and gets first pick in the upcoming draft. The 22 coaches, scouts, and executives are voting between 5 candidates: Allen Byers, Castillo, Dixon, and Evans. Here are their preference ballots.

	2 Voters	6 Voters	4 Voters	1 Voter	1 Voter	4 Voters	4 Voters
Allen	1	2	2	3	3	2	5
Byers	4	1	1	2	4	5	4
Castillo	3	3	5	1	1	4	2
Dixon	2	4	3	4	2	1	3
Evans	5	5	4	5	5	3	1

After all tallying is done, Allen has 3 points, Byers 2.5, Castillo 2, Dixon 1.5, and Evans 1. So Allen is the winner with *Copeland's Method*.

3.8 Approval Voting

With the approval voting method, voters indicate their approval or disapproval of each of the candidates. A ballot in an approval vote lists the candidates and voters will check off all the candidates of whom they approve. The winner is the candidate with the highest approval count.

For example, suppose there are 5 candidates for chair of a department at a university. There are 11 professors in a department, and the department uses approval voting.

	1 Voters	1 Voters	3 Voters	1 Voter	2 Voter	1 Voters	2 Voters
Belding			✓			✓	✓
Critchlow	✓		✓	✓		✓	✓
Eck					✓		✓
Oaks				✓			
Vaughn		✓			✓	✓	✓

Here, Belding earns 6 votes, Critchlow earns 8, Eck 4, Oaks 1, Vaughn 6. Critchlow wins.

4 Fairness Criteria

Certain criteria can be expected, or at least desired from an electoral system. In the spirit of democracy, these criteria try to mathematically capture what the ideals of a democratic election are. We will discuss four of them.

4.1 Non-dictatorship

The property of *non-dictatorship* is satisfied if there is no single voter with the individual preference order P, such that P is always the societal winning preference order. In other words, the preference of an individual should not always prevail over others. In a blind voting system with more than two-people, all voters are non-dictators.

4.2 Unrestricted Domain

For any set of individual voter preferences, the social welfare function with unrestricted domain should yield a unique and complete ranking of societal choices, that completely ranks the societal preferences, and does so deterministically.

4.3 Independence of irrelevant alternatives (IIA)

The societal preference between x and y should depend only on the head-to-head preference between x and y. Changes in an individual voter's rankings of candidate z should not a affect the societal position of x and y, even if candidate z drops out of the race.

4.4 Pareto efficiency (unanimity)

If every individual prefers a certain option to another, then so must the resulting societal preference order. This, again, is a demand that the social welfare function will be minimally sensitive to the preference prole.

4.5 Arrow’s Impossibility Theorem

THEOREM: When voters have three or more distinct alternatives (candidates), no ranked voting electoral system can convert the ranked preferences of individuals into a community-wide (complete and transitive) ranking while also meeting the four criteria.

Even still, Kenneth Arrow said “most systems are not going to work badly all of the time. All I proved is that all can work badly at times.” As Arrow’s theorem only applies to preferential (or ranked) voting systems, some political theorists have admired the complexity of cardinal voting systems, where candidates are given a score, not just a rank. Some argue that ratings are fundamentally invalid, because meaningful interpersonal comparisons of utility are impossible. This was Arrow's original justification for only considering ranked systems, but later in life he stated that cardinal methods are "probably the best".

5 Peer Response

I enjoyed reading everyone’s responses to my paper, there was a lot of helpful constructive criticism in there, in addition to generally positive feedback. My humble apologies for going over the clock, I seem to try and pack too much in always. One failure of mine is the common takeaway from this information that there is no perfect voting system, that is not the mathematical result of Arrow’s Impossibility Theorem, and even though I pointed out that this theorem only describes ranked voting methods, which I think people got, and stressed that while the criteria are rather sound, they are by no means the perfect representation of democratic voting themselves, that the layman interpretation of the theorem needs to be more restricted than the often misquoted “there is no perfect voting system.” I should have been more cautious against that misinterpretation. I am happy to see that people did not mind the many definitions that a presentation like this requires, so that was good, and people liked the topic and thought it was informative and engaging, which is great to hear.

6 Conclusion

In all, we discussed a small survey of election theory, and talked about many of the fundamental criteria to determine the fairness of elections themselves. We considered preferential, approval, and even a little bit of ordinal voting systems, discussed both the mathematical and practical limitations of voting systems, and I think most everyone came away with a more rigorous understanding of the important behind-the-scenes math that our society elects its leaders with, political or otherwise. I think especially instructive was seeing how the same number of preferential ballots could elect different leaders under the different voting mechanisms. People who came to this lecture should be able to come away from this presentation with a better understanding of the math behind democracy.

References

Arrow, K.J., 1950, “A Difficulty in the Concept of Social Welfare”, *Journal of Political Economy*, 58: 328–346.

COMAP. *For All Practical Purposes: Mathematical Literacy in Today's World*. Macmillan Learning, 2015.