MATH 400: SANGAKU, JAPANESE TEMPLE GEOMETRY

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Abstract. Geometry has been a topic of fascination for humankind since the earliest known developments in mathematics. The study of geometry has spanned thousands of years across different cultures throughout the world. For my first MATH 400 presentation, I aim to expose the Japanese cultural phenomenon of Sangaku - mathematical tablets that proved theorems an propositions in Euclidean geometry and were hung in Shinto shrines and Buddhist temples throughout the Edo Period (1603-1868). This paper is organized into four parts. In Part 1, I will discuss why I chose this as a relevant mathematical topic and how it fits the goal of the course with regards to "Mathematical Connection." In Part 2, I will provide the historical background of the Edo Period of Japan, the unique Japanese way of doing mathematics known as Wasan which led to the creation of Sangaku, and discuss the significance of Sangaku. In Part 3, I will provide examples of Sangaku Problems with proofs using modern mathematical notation. In Part 4, I will close with a reflection on my presentation to the class and the feedback I received from other students, as well as lessons learned for my next presentation.

1. Mathematical Connection

1.1. **Topic Selection and Introduction.** When deciding on a topic that would be interesting to present to the MATH 400 class I knew I wanted to expose a subject in mathematics they were not already familiar with. Fortunately, through my mathematics education here at William & Mary I was familiarized with the niche study of Sangaku problems. Sangaku problems were something that peaked my curiosity because of my interest in geometry as well as my cultural connection as a half-Japanese citizen. Furthermore, on the theme of "mathematical connection," I thought it would be a perfect subject to highlight the intersection of history, art, culture, and linguistics with the field of math to show how math can be connected to via a multitude of interdisciplinary subjects. Another advantage of Sangaku problems are that they have a great variance in the level of difficulty in the types of problems offered. You can start with simple elementary shapes and problems that could be taught to grade-school children (as was done in the Edo period with Japanese students) but can quickly ramp up to medium level problems such as the ones offered in this paper to extremely difficult problems that have been left unsolved (see Gion Shrine problem). The ultimate goal for my presentation was to introduce this unique cultural phenomenon and outline the characteristics of Japan during the Edo period to highlight this distinctive mathematical movement.

Date: March 3, 2020.

2. Sangaku

2.1. The Edo Period and Genroku Renaissance. To first understand Sangaku, it is necessary to have an understanding of Japan during the Edo period (1603-1868). During the Edo period, the Tokugawa Shogunate isolated Japan from the rest of the world for nearly 300 years until the 19th century with a foreign policy known as *sakoku* (closed country). This period of isolation eliminated foreign conflicts and was known as the "Great Peace" or "Genroku." Japanese culture flourished during this time and was seen as a renaissance period. It was during this period that distinctively Japanese cultural practices gained significance and were celebrated throughout the country. Within poetry, there was the growth of Haiku. Within theater, Kabuki and Noh (classical forms Japanese dance-drama) flourished in theaters. In art, Japanese flower arrangements and garden architecture were ascendant. Finally, in mathematics, there was Sangaku. [5][10]

2.2. Wasan - Japanese Mathematics. Due to the policy of sakoku, the Tokugawa Shogunate enforced a ban on thousands of Western books, including those in astronomy, geography, and mathematics. As a result of the reduction of the number of works in math, the Japanese developed their own works on the subject. This sprung into development of the uniquely Japanese way of doing mathematics known as Wasan. Western mathematics was not introduced to the Japanese school system until the beginning of the Meji Period (1868-1912). Prior to this, Wasan - literally meaning Japanese arithmetic - was the defining way of doing mathematics among Japanese intellectuals. Characteristic of Wasan is its use of *Kanbun*, an archaic form of Japanese that is closely tied to Chinese, rather than the Arabic numbers and Latin letters familiar in modern mathematics. [6][10]

Figure 1. An example of a proof for a Sangaku problem using Wasan (Katayamahiko Temple Problem in Section 3.3) [6]

2.3. **Sangaku as a Religious Offering.** In japanese *san* translates to calculation and *gaku* translates to plaque, thus *sangaku* literally meant "mathematical plaque." Sangaku was likely influenced by the pre-existing tradition of hanging tablets in Shinto shrines known as *ema*, where *e* means picture and *ma* means horse. Originally, people donated sacrificial horses to shrines for good favor; over time these donations became pictures of horses and later into intricate artworks of battles, peoples and landscapes. Ema was hung for hundreds of years at Shinto shrines prior to the Edo Period as an offering to the "Kami" (Japanese spirits or gods). However Ema were importantly distinctive from Sangaku in that they never included mathematical ideas. Thus, sangaku likely began as a distinct "mathematical ema" that was inspired but completely new from the ema tradition. [6]

FIGURE 2. Sangaku displayed under the roof of a temple in Fukushima [9]

2.4. **The Spread of Sangaku.** The oldest surviving Sangaku dates from 1683 and had been found in a temple in the Tochigi prefecture in Japan. Over the next 200 years, Sangaku appeared in temples and shrines throughout japan with about two-thirds in Shinto shrines and one-third in Buddhist temples. Today, roughly 880 Sangaku remain where as at least 1,738 have been lost or destroyed according to contemporary mathematics texts. The majority of the presenters of sangaku were members of the samurai class who turned from warrior to mathematician during the peaceful Edo period, however there have been cases of women and children who have also created Sangaku. There were no formal colleges or universities in Japan, instead teachings were carried out in private schools known as *jukus* and in shrines and temples. It is this close connection with religious centers as places of education, along with its role as a religious offering, that explain why Sangaku were only found in shrines and temples. [5][10]

2.5. **The Structure of Sangaku.** In general, Sangaku tablets included an artistic representation of the problem posed and supporting text in the following format:

(1) **Problem:**
(2) **Answer:**
(3) **Explanation:** (*Jojutstu* Formula)

Most Sangoku, due to the constriction of space on the tablet, did not include a proof of how to complete the posed question. Instead, it is often considered that Sangoku Tablets were posed as a challenge and left to the reader as an exercise. As substitution for a rigorous proof, the Jojutsu (formulas) used in the proof were offered. Jojutsu were unique to Wasan, but included commonly understood results such as the Pythagorean theorem. [7]

3. Sangaku Problems

In the following section, I present what I believe to be some representative problems of Sangaku that were created and hung in different shrines and temples during the Edo Period. With the exception of the Katayamahiko Problem, the original tablets of the following problems have been destroyed but records of them were discovered in the journals and diaries of mathematicians of the age. Sangaku problems ranged significantly in difficulty from problems that were elementary and postulated by young teenagers to extremely challenging and would require several pages of rigorous proof.

3.1. **Akahagi Kannon Temple Problem.** It is often cited that Sangaku was so commonplace that even women and children participated in its creation. Some academics, however, hold reasonable doubt as to how widespread the practice actually was. The following problem however, was proposed by thirteen year old Sato Naosue and hung in 1847 in the Akahagi Temple in Ichinoseki city. The modern translation have been supplied by Fukugawa, Rothman and the following illustrations and proof are by Bogomolny. [5][2]

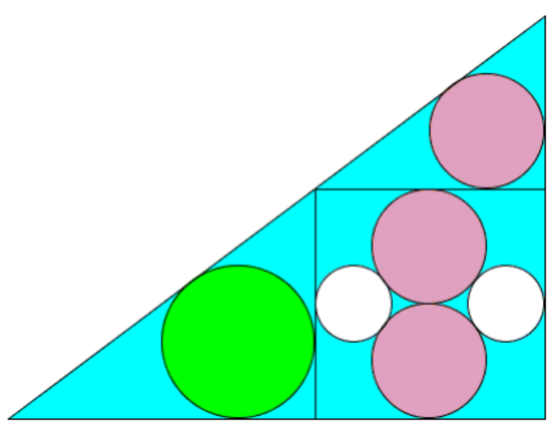

Figure 3. Akahagi Kanon Temple Problem [2]

(1) **Problem:** Two pink circles of radius r and two white circles of radius t are inscribed in a square, as shown. The square itself is inscribed in a large triangle and, as illustrated, two circles of radii r and R are inscribed in the small triangles outside the square. What is the relation between R and t?
(2) **Answer:** $R = 2t$.
(3) **Explanation:** (Provided by Bogomolny [2])

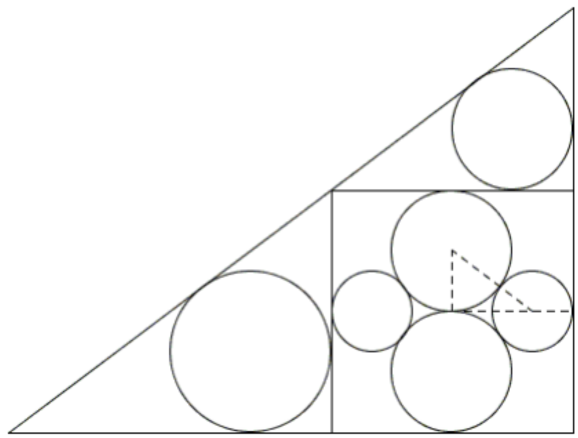

FIGURE 4. Akahagi Kanon Temple Problem Solution [2]

Proof. Obviously, the side of the square is of length $4r$. One application of the Pythagorean theorem (in the dashed triangle) shows that $r = 3t/2$. Use the Pythagorean theorem again in the small upper triangle:

Let a be the small length of that triangle and c its hypotenuse. On the other hand, the area S of the small triangle can be computed in two ways:

$$2S = r(a + c + 4r) = 4r \cdot a,$$

such that $a + c + 4r = 4a$; from which $c = 3a - 4r$. Substitute that into the Pythagorean identity - $a^2 + (4r)^2 = c^2$ - to get the equation

$$a^2 + 16r^2 = (3a - 4r)^2,$$

solving which gives $a = 3r$.

The triangle turns out to be the famous 3-4-5 or Egyptian triangle with the short side equal to $3r$. From the similarity of the three triangles, all of them have the proportions 3-4-5 which leads to $4r = 3R$.
And finally, $R = 4r/3 = 2t$. □

3.2. **Meiserinji Temple Problem.** Several sangaku involve problems with inscribed circles. Most of these were easily solved with the Pythagorean theorem and were probably more works of art than mathematics. The following was created by a woman, Okuda Tsume in 1865 at Meiserinji Temple in Ogaki City, 1865. As above, the modern translation has been provided by Rothman, Fukugawa and the following illustrations and proof are by Bogomolny. [4][1]

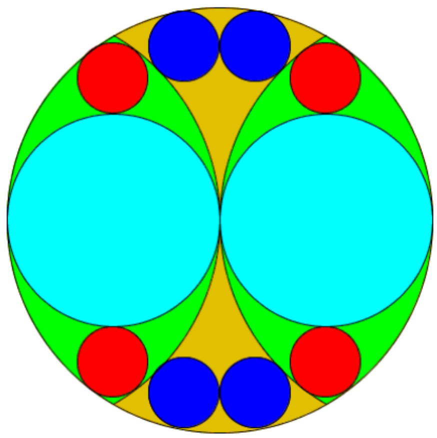

FIGURE 5. Meiserinji Temple Problem [1]

(1) **Problem:** In a circle of diameter $2R$, draw two tangent arcs of radius R, and then ten inscribed circles, two of diameter R; four red of radius t and four blue of radius t'. What is the relation between t, t' and R?
(2) **Answer:** $t = t' = R/6$.
(3) **Explanation:** (Provided by Bogomolny [1])

Proof. The solution is based on the following diagram

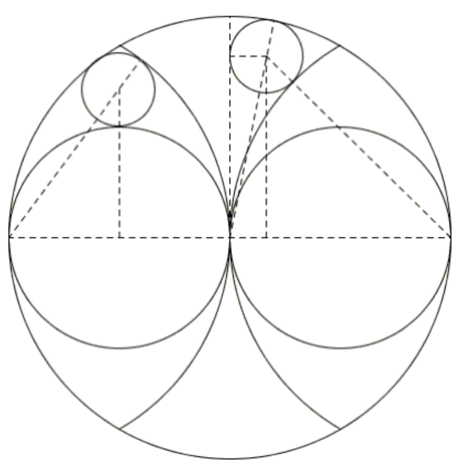

FIGURE 6. Meiserinji Temple Problem Solution [1]

For t we consider the left dashed triangle:

$$(R/2)^2 + (R/2 + t)^2 = (R - t)^2.$$

For t', consider the two right dashed triangles with a common altitude. Applying the Pythagorean theorem twice and equating the altitudes gives:

$$(R - t')^2 - (t')^2 = (R + t')^2 - (R - t')^2.$$

The two equations are easily manipulated into the desired identities. □

3.3. **Katayamahiko Temple Problem.** This problem is the fourteenth of sixteen problems from a single sangaku dedicated in 1873 to the Katayamahiko Shrine in the town of Osafune. Hosking provides a direct translation of the problem from Japanese to English as well as how to solve the problem using the traditional Wasan method of Tenzen Jutsu. [6]

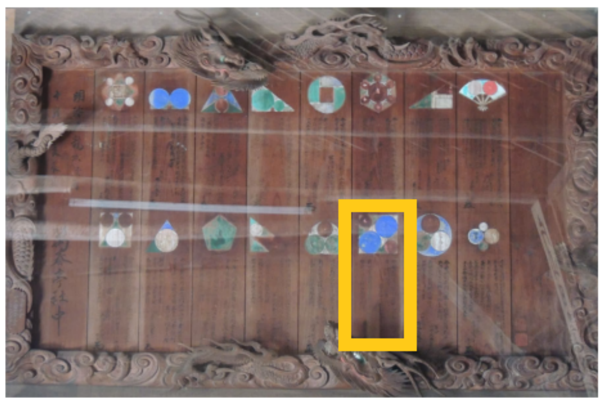

FIGURE 7. Katayamahiko Temple Sangaku (Problem 14 highlighted in yellow) [6]

The following has been rephrased in more understandable English by Fukugawa, Rothman and the proof and illustration are provided by Hosking. [5][6]

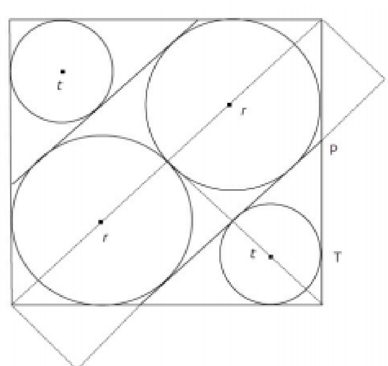

FIGURE 8. Katayamahiko Temple Problem [6]

(1) **Problem:** Two circles of radius r are inscribed in a square and touch each other at the centre. Each of two smaller circles with radius t touches two sides of the square as well as the common tangent between the two larger circles. Find t in terms of r.
(2) **Answer:** $t = 0.585786r$ (original answer $t = 0.585r$)
(3) **Explanation:** (Provided by Hosking [6])

Proof. We see that, on the one hand, the length of the central diagonal is $2r + 2\sqrt{2r}$. On the other hand, it is also equal to $2PT + 2r$.
However $PT = t(1 + \sqrt{2})$.
Equality the two expressions gives:

$$t = \frac{\sqrt{2}}{\sqrt{2}+1} r = (2 - \sqrt{2})r = 0.585786r$$

□

3.4. **Gion Shrine Problem.** The Gion Shrine problem may be the most famous or notorious of Japan's Sangaku problems. Geometrically simple and elegant, the solution requires extremely intense algebraic calculation. Since it was first presented in 1749, the Gion Shrine problem has puzzled people for more than 200 years and received significant attention from several prominent Japanese mathematicians. Reyna, Clark, Elkies have reworded the problem as below: [8]

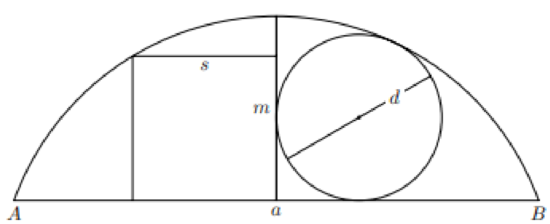

FIGURE 9. Gion Shrine problem [8]

(1) **Problem:** We have a segment of a circle. The line segment m bisects the arc and chord AB. As shown, we draw a square with side s and an inscribed circle of diameter d. Let the length $AB = a$. Then, if

$$p = a + m + s + d \quad \text{and} \quad q = \frac{m}{a} + \frac{d}{m} + \frac{s}{d},$$

find a, m, s, and d in terms of p and q.

The problem was first proposed by Tsuda Nobuhisa in 1749 on a sangaku of the Gion Shrine in Kyoto. He provided a "solution" of the problem first by deriving a polynomial of degree $1024 = 2^{10}$ from whose roots one could derive the results. The next major breakthrough came from mathematician Nakata, who reduced the necessary polynomial degree to 46. It was Ajima Naonobu in 1774, however, who gained the most praise by reducing it to a 10 degree polynomial using no geometric techniques more sophisticated than the Pythagorean Theorem [8]. According to Reyna, Clark, and Elkies,

> "With a great deal of algebraic persistence, [Ajima] is able to manipulate a few basic geometric relations into a system of high degree equations in a and d. A clever substitution yields four cubic equations in a single variable X whose coefficients are given in terms of a, p, and q. This can be viewed as a homogeneous linear system with nontrivial solution $(X^3, X^2, X, 1)$; any such system must have determinant zero. Ajima then uses a technique equivalent to Laplace's method of cofactor expansion (c. 1776) to arrive at a polynomial equation of degree 10 in a, which requires nearly a full page to write out completely. It should be noted that, because of *sakoku*, Ajima (1732–1798) may not have even heard of Laplace (1749–1827), and likely was unaware of his results." [8]

For his effort in reducing the complexity of the equation of the Gion Shrine problem down to 10 degrees, Aijima gained great renown as a mathematician. This feat, along with his other significant contributions in the time, made him considered as the greatest Japanese mathematician of the eighteenth century. [5]

Modern solutions to the Gion Shrine Problem have been suggested however they require mathematics not known in Japan at the time the problem was written. Reyna, Clark and Elkies proposed one such modern solution in 2015 (also a degree-ten equation), as well as results in the existence/uniqueness of solutions and proved that the problem has no rational solutions. No analytic solution has yet been found, thus the problem is still considered unsolved to this day. [8]

4. Presentation Reflection

4.1. **Personal Reflection.** My objective for this presentation was to introduce this unique cultural phenomenon and discuss the interesting intersection of culture, history, linguistics and mathematics that is present in studying Sangaku problems. Some of the key points I outlined included the characteristics of the Edo period of Japan, the unique Japanese method of mathematics known as Wasan, and the Sangaku problems themselves. I thought that explaining the historical and cultural context of Sangaku helped aid my presentation and made the topic more interesting. Some lessons learned for my next presentation would perhaps be to spend a little more time explaining the proofs of the different example problems, however with my limited time I did not want to spend too much time on proofs but rather create intrigue and curiosity in the audience to seek these problems later if they were interested.

4.2. **Peer Feedback.** After reading through my peer feedback from the discussion board I got the sense that my presentation was generally well-received. Several students cited their interest not only in the mathematics but the connection to history that made them more engaged in the topic. For future presentations, I will remember this feedback and make sure to include interdisciplinary areas such as history to enrich the mathematical content of my lecture.

4.3. **Related Topics for Further Research.** Some of my peers gave other areas for possible future research. One of these included further expansion into solvable and unsolvable problems, inspired by the discussion on the Gion Shrine Problem. Along the same theme, another student recommended research into the famous Hilbert Problems that were unsolved at the time of their proposition. Professor Li also recommended further study into different proofs of the Pythagorean Theorem and even gave one conceived by a Japanese mathematician as an example. Lastly, I would personally like to do more research into Islamic tilings. While different from Sangaku, Islamic tilings are intricate artistic patterns used throughout much of Islamic architecture. Their geometry lends itself to some interesting mathematics and similar to Sangaku they provide a unique insight into the cultural background of where they were created.

References

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