The Four Color Theorem

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1 Introduction

This paper will take examine the mathematical theorem known as the Four Color Theorem. We will begin by outlining hate history of the theorem, the numerous insufficient proofs presented throughout history, and finally how the theorem was ultimately proved. We will then discuss the implications of this theorem on the mathematical community, and its important historical significance.

2 History and Setup

The Four Color Theorem dates back to 1852 and a mathematician named Francis Guthrie. One day, Guthrie decided to color in a county map of England and challenged himself to see if he could color in the map using only four colors. Guthrie was able to do this, and then began to wonder if it was possible to color in any map using just four colors. Guthrie posed this question to his brother, Frederick Guthrie, a mathematics student at University College London. At university, Frederick Guthrie was a student under Augustus De Morgan, most notable for De Morgan's Laws, who became interested in this question as well. The question was published by De Morgan in a London magazine in 1854, and then again in 1860.

This theorem, however, would go unproven for over 100 years.

From this point on we will discuss how the theorem was proved, and its implications for the mathematics world. We will be framing much of this paper, however, from an educational standpoint. The flow of this paper will walk you through not only how the theorem was proved, but will follow a general guideline of a lesson plan on how one might wish to teach this concept in a classroom setting. This paper has been given as a 30 minute presentation which followed the order of this paper rather closely.

3 How to Prove the Theorem

The first thing we should discuss when approaching this problem is some rather rudimentary graph theory. In graph theory, the network maps are comprised of three features: faces, edges, and vertices (or nodes). Let us take a look at an arbitrary possible map configuration. Note, this map does not look like many maps that we are familiar with, but it does represent a cleaner possible configuration for a map.

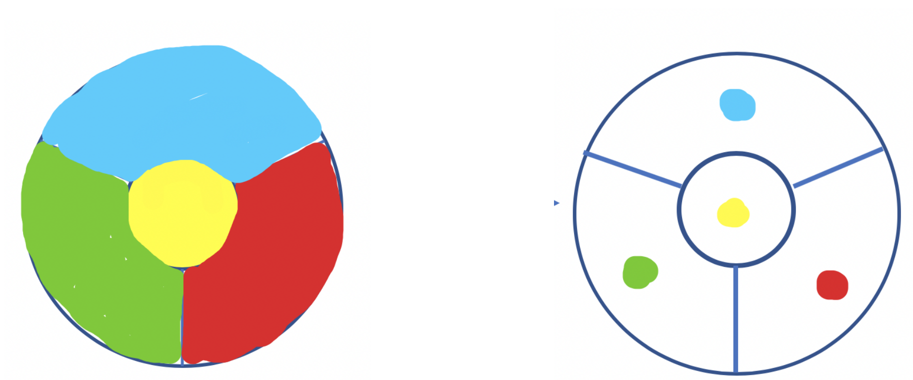

Figure 1: Example Map 1

When looking at our initial map to the left, we can see that simply coloring in a small dot is sufficient and cleaner than coloring in the entire map. The real reason we wish to color in our map this way, however, is because of how graph theory is going to come into play. When looking at those four colored dots, let us draw lines connecting every "country" that borders each other. The resulting configuration would look like this.

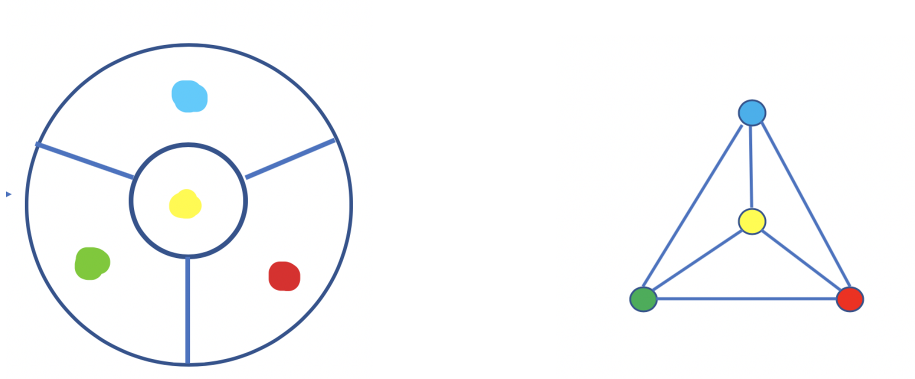

Figure 2: Network Graph for Map 1

Drawing our maps like this is not only cleaner and easier to understand, but also allow us to conceptualize the possible map configurations we must consider. One important use of these configurations is to determine when two maps are the actually the same, even though they may look different. For example,

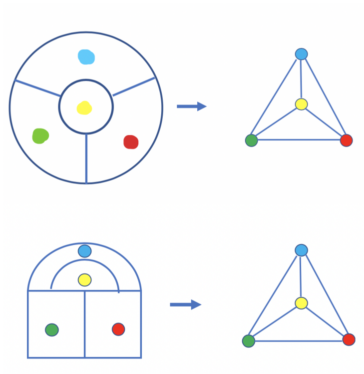

Figure 3: Network Graph for Map 1

In drawing our maps using graph theory we are able to see duplicate map configurations, thus minimizing our amount of work. Furthermore, these configurations are useful in determining which network maps are valid maps and which are impossible configurations in the real world. Just because you can draw a network map for some configuration does not mean that that configuration is a valid map. As seen in *Figure 2*, we have four nodes connected with edges designating bordering countries. We can also note that none of these edges are overlapping. This is an important characteristic, as any map in which two edges overlap is not a valid map. For example,

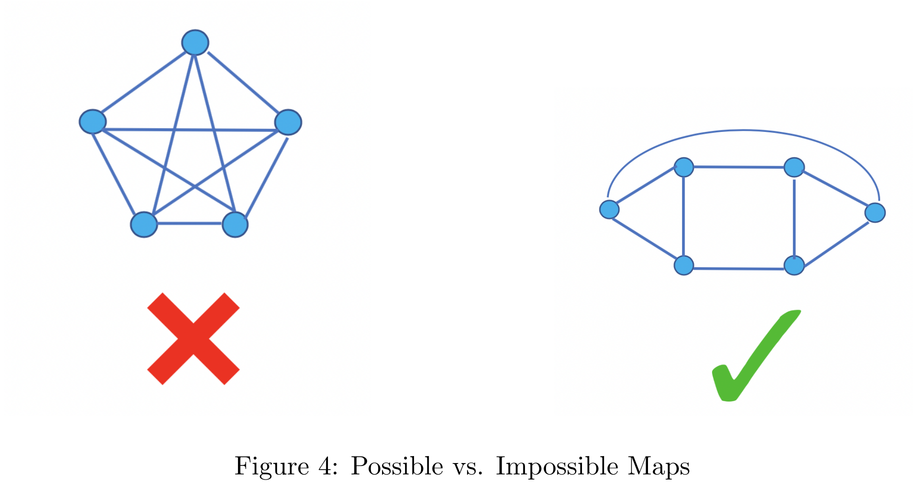

Figure 4: Possible vs. Impossible Maps

In the image on the left, we see numerous edges of our graph that overlap.

If you try and conceptualize this graph as a map, it is simply impossible. That does not mean, however, that all graphs that might have crossing edges are impossible. The image on the right could have been drawn with an edge connecting the leftmost node and the rightmost node going straight across. However, what is expressed in the image is the ability to "untangle" those overlapping edges. If a network map with overlapping edges can be redrawn without those edges overlapping, it is still a valid map. However, if there is no possible way to untangle those overlapping edges than it is not a valid map and therefore is not a consideration in this paper or the theorem in general.

Also with the use of graph theory we can determine certain elements that *every* map must have. Using graph theory we know that every map must contain at least one of these network configurations within it:

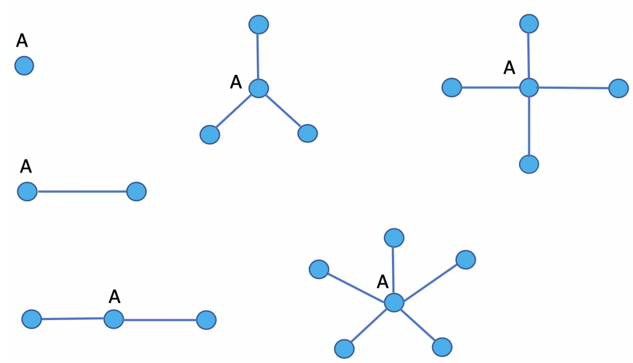

Figure 5: Necessary Elements

Knowing that at least one of these network configurations exists within every possible map, we can now begin to prove the Four Color Theorem. But before we can prove the Four Color Theorem, let us try and prove a simpler problem: the Six Color Theorem.

4 The Six Color Theorem

When approaching the Six Color Theorem, let us first define what that means. The Six Color Theorem states that every possible map can be colored using only 6 colors. If we assume that this theorem is false, then that would mean that there exists some map that requires the use of 7 colors. Let us imagine the fictional smallest possible map that requires the use of 7 colors. This imaginary map is shown in the figure below.

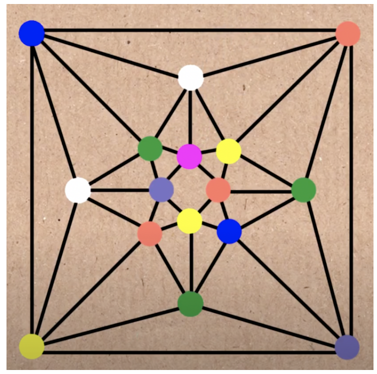

Figure 6: Smallest Possible Map Requiring 7 Colors

Now, knowing what we have discussed about graph theory thus far, we know that this imaginary map will contain at least one of the elements shown in *Figure 5*. Let us take one such element out, as shown in the figure below.

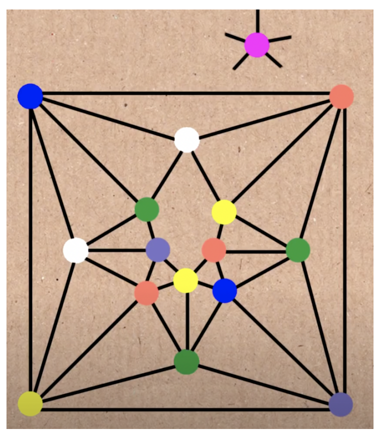

Figure 7: Smallest Possible Map Requiring 7 Colors, Minus One Country

As the map in *Figure 6* was the smallest possible map that required 7 colors, our new map in *Figure 7* can be colored using only 6 colors. Now, let us place that element we took out back into the map. As whatever element we took out of the map has five or fewer countries connected to it, even if five different colors are used for each of its neighboring countries, there will still be a sixth color leftover for our country. Therefore, when we place this country back in the map, it is still colorable using only 6 colors. We have thus proved that all maps are 6 colorable.
The same logic applies for the Five Color Theorem, with one exception. If the arbitrary element taken out of the map has 5 neighbors, you will need to recolor the map such that all five of the neighbors are not different colors. Once you

do this, however, placing the remaining element back into the map will yield a map that is still 5 colorable, thus proving the Five Color Theorem.

5 How They Finally Proved the Four Color Theorem

Although numerous proofs were presented throughout history, no one was able to provide a sufficient proof until 1976. In 1976, Kenneth Appel and Wolfgang Haken of the University of Illinois announce that they had proved the theorem with the assistance of computers. Appel and Haken's proof was comprised of two main concepts: unavoidable sets and reducible configurations. Unavoidable sets are a set of configurations such that every map contains at least one of the configurations from the unavoidable set. Additionally, if a map contains a reducible configuration, then the map can be reduced to a map with fewer regions so that if the smaller map can be colored with four colors then so can the bigger map. Appel and Haken produced an unavoidable set containing 1,936 reducible network configurations. This unavoidable set was tested using a dynamic programming algorithm, and was the first mathematical proof ever solved with the assistance of a computer.

As this theorem was proved by checking every possible case using a computer, many in the mathematics world were unsatisfied with this solution. Even though they were able to proof the theorem, we did not learn anything significant as to why it works or what broader implications it may have.

6 Conclusions and Future Research

The Four Color Theorem is a fascinating theorem that challenges how we think about mathematics. This theorem went unproved for over 100 years, and even now that it is proved we still do not entirely know why it works.

Further research could explore more modern uses of technology and computers in mathematics, and the role that AI is and will play in that. Today, there are software programs capable of checking the logic of mathematical proofs. What does this increasing role of technology in mathematics mean for the future of how we solve problems? Should we blindly trust the proofs and outputs of these computers if we do not have the means to check the math ourselves? All of these questions show why this theorem is so interesting and so controversial to many.

References

[1] "Four Color Theorem." Wikipedia, Wikimedia Foundation, 12 May 2020, en.wikipedia.org/wiki/Four_color_theorem.

[2] The Four Color Theorem, 2017, people.math.gatech.edu/ thomas/FC/fourcolor.html.

[3] Rogers, Leo. "The Four Colour Theorem." NRICH, nrich.maths.org/6291.