Non-Euclidean Geometry - Applications of the Impossible

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ABSTRACT

Euclid's *Geometries* was the definitive mathematics textbook for over two thousand years, but its fifth postulate troubled many thinkers over its long tenure. In the 1800s, mathematicians began to question its validity, resulting in the rise of non-Euclidean geometry, a system of axioms and theorems that seems to fly in the face of the concept of measure we know and understand today. However, its predictions proved prophetic in explaining the nature of spacetime, as Einstein formulated in the early 20th century. This report aims to explain the history and reasoning behind non-Euclidean geometry, and explore its application in general relativity.

Introduction

Ingenuity and innovation is the beating heart of scientific inquiry; without it, the scientific method has no impetus to promulgate itself, and our understanding of the natural world lies fallow. Mathematics is no stranger to this fact, and the continued march of mathematical research depends on the dynamic search for new ideas that challenge the established order. Nowhere is this more clear than in the revolution that shook the foundations of empiricism itself: the development of non-Euclidean forms of geometry in the early to mid-19th century CE. This bold new theory had profound implications for math, physics, philosophy, and all the natural sciences, and would pave the way for new revolutionary discoveries of the 20th century that would change our understanding of the universe forever.

Euclidean geometry

The text *The Elements* was the final, authoritative textbook of geometry for nearly two millennia, and for good reason: its axiomatic construction and sound conclusions are astonishingly clear and precise, and hold up sturdily even to modern analysis. While its author Euclid, writing in the 4th century BCE, can't lay claim to all of its contents as his and his alone, his was certainly the decisive amalgamation that led to the development of nearly all modern geometry (Trudeau, 5). Euclid based his 13-volume work on several "common-sense" assumptions and 5 geometrical axioms, translated as follows:

1. To draw a straight line from any point to any point
2. To produce a finite straight line continuously in a straight line
3. To describe a circle with any center and distance
4. That all right angles are equal to one another
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles (Trudeau, 39-42)

From these 5 axioms, Euclid sets out to prove most of the geometry taught in modern middle and high schools - congruence and area of triangles, rectangles, and squares, as well as the Pythagorean Theorem, to name a few. This treatise was for centuries regarded as the foundational truth from which all the sciences sprang from, and philosophers praised it as the most elegant and purest form of knowledge known to mankind (Trudeau, 106).

Confronting the Fifth Postulate

However, the fifth postulate left some scholars (both of antiquity and in the modern day) feeling ill at ease, since it seemed less intuitive and immediately obvious than the prior four. If geometry was the pinnacle of human knowledge, then why must it rest on such an "ugly" foundation as this jumble of formalism? For centuries mathematicians and philosophers availed themselves of new ideas to prove it from the previous four, only to come up empty-handed. Some, like John Playfair in the 18th century, created substitutions for this postulate, such as "Through a given point not on a given straight line, and not on that straight line produced, no more than one parallel straight line can be drawn," which is more digestible and is used in mathematics textbooks to this day (Trudeau, 128). But these substitutions are logically equivalent to the fifth postulate, and as such don't resolve the unease associated with its existence. This "hole" in mathematics wasn't patched in a rigorous way until the dawn of the 19th century, when the foundations of geometry were shaken by new insight.

Non-Euclidean geometries

Very near to the year 1818, one German jurist and mathematician named Ferdinand Schweikart assembled a collection of theories about the fruitless quest to prove postulate 5 and the possibility of other alternative paths. These he sent to Carl Gauss, a German mathematician of great renown, who found it enlightening and "almost as if he had written it himself." (Bardi, 127) Gauss, perhaps as early as 1813, had come to similar conclusions during his own musings on the topic (Trudeau, 157). However, neither of these mathematicians published their findings, for the fear of the disruption it would cause in the mathematical world; as Gauss put it, "But the wasps, whose nests you stir up, will fly around your head." (Bardi, 124). These theories didn't reach publishing houses until 1830, when Russian mathematician Nikolai Lobachevsky published his treatise, titled *On the Principles of Geometry*, which focused on expanding the rules that would take place on non-Euclidean geometry (Trudeau, 158). Gauss' concern seemed to ring true, and Lobachevsky's ideas were ridiculed by his peers. Just two years later, Hungarian mathematician Janos Bolyai published an appendix to his father's paper entitled *Absolute Science of Space* which outlined a similar framework (Bardi, 162). While these works originally went ignored by the mathematical community, by the mid-1800s their significance came to be recognized and the non-Euclidean revolution was in full swing.

In these seminal works, authors took the fifth postulate and replaced it by a negation; namely, that having interior angles less than 180 degrees doesn't guarantee intersection. The consequences of accepting this idea are multitudinous: in this formalism, a triangle is composed of angles that sum to less than 180 degrees, and quadrilaterals' interior angles sum to less than 360 degrees. Proofs of these statements can be found in Appendix A. It was a radical departure from the previously unquestioned reality of Euclidean geometry, and seemed counter to all common-sense; after all, any triangle I draw seems to have interior angles equal to 180 degrees. But in spite of this seeming impracticality, this new formalism would pave the way for new developments in practical applications, and reshape modern physics.

Einstein's Theory of Relativity and Minkowski space

In 1919, a solar eclipse allowed scientists to measure the location of a star very near to the sun, and found it somewhere it shouldn't be (Bardi, 208). This stunning discovery vindicated the work of German physicist Albert Einstein and his theory of general relativity, a revolutionary new way of understanding space, time, and the laws Newton promulgated centuries before. In his formalism, space is not Euclidean, but rather warps into non-Euclidean geometry in the presence of massive bodies (Halstead, 2.1). Since a straight line between two points in this space is no longer a Euclidean "straight" line, objects follow curved paths around massive bodies - including the light from distant stars. To understand how this warped space functions, a new metric for measuring space was formulated by Hermann Minkowski in which length is not invariant between reference frames, but rather the "spacetime interval" - a quantity that incorporates both spatial distance and time - is invariant (Halstead, 1.2). It can be defined using the equation

$$\Delta\sigma^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

where the variable σ is known as the "proper distance." (Halstead, 1.2) Encoded into this formalism is the recognition that the speed of light is constant in all reference frames, and time is distorted for a body moving at speeds close to the speed of light. While Minkowski space is equipped to model massless objects (which is the case in Einstein's theory of special relativity), Einstein's theory of general relativity incorporates massive bodies into the geometry, which arise from Einstein's field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \text{ (Halstead, 2.1)}$$

In this formulation, $G_{\mu\nu}$ is a system involving $g_{\mu\nu}$ and $T_{\mu\nu}$ is a representation of the distribution of matter in space. Λ, the so-called cosmological constant, is used to account for the overall geometry of the universe. In total, this is a system of 10 coupled, nonlinear, second-order partial differential equations - a daunting task to solve and beyond the scope of my ability - but the end result is discovering $g_{\mu\nu}$, the metric that modifies the spacetime interval of Minkowski space (Halstead, 2.1).

Schwarzschild geometry

Solving the field equations for a single, non-rotating mass (such as a planet or star) produces the following warping of Minkowski space:

$$d\sigma^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \text{ (Donis)}$$

This is written in spherical coordinates, and the constants G and c are assumed to be equal to 1. In the case where $r \rightarrow 2M$, the Minkowski space becomes increasingly distorted, and geodesics (straight-line paths through space) converge as $r \rightarrow 0$ (Donis). This is exactly the prediction of elliptical geometry - that all lines that are initially parallel eventually meet. In this system, space and time itself follow non-Euclidean geometries, and are perceived as a gravitational field. Einstein's discovery catapulted him to worldwide fame, and his field equations paved the way for the discovery of black holes, dark energy, and gravitational waves. Today, LIGO detects oscillations in spacetime created by black holes billions of years in the past ("Gravitational Waves").

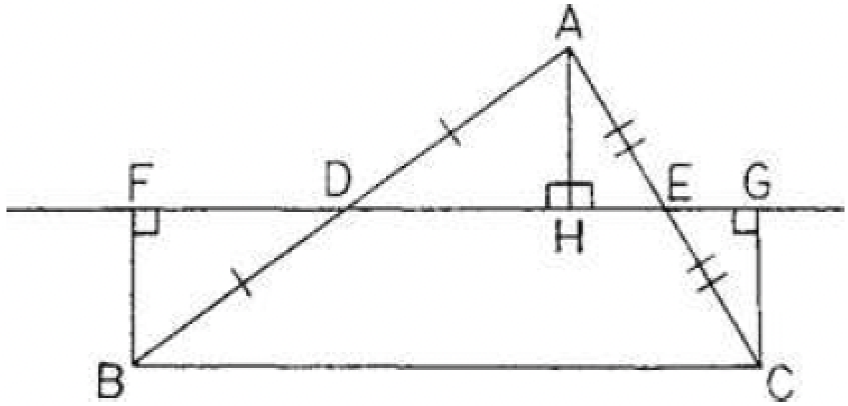
Philosophical implications and conclusion

It isn't immediately obvious to the senses that geometry that rejects Euclid's 5th postulate would ever have a physical counterpart in our real world, yet just a century later discoveries justifying its existence came to light. Today, non-Euclidean metrics of distance like Minkowski space and its role in the Einstein field equations are utilized daily by astrophysicists, rocket scientists, and even GPS satellites, and are necessary for the continued functioning of many of our electronic devices. The necessity of daring, bold mathematicians who aren't afraid to disrupt the status quo is made obvious by these leaps forward that enabled the modern world to develop. It begs the question, what commonly-held beliefs need to be challenged and reimagined for future progress to be possible? What radical ideas are seen as ridiculous that actually contain sparks of genius? The scientific method enables us to step forward gradually, but occasionally it must be salted by a visionary unafraid to get their feet wet in uncharted waters.

Discussion

Following my presentation on the 21st, our class discussed the implications of the discovery of non-Euclidean geometry and its relationship with general relativity in the sciences, in math, and in the field of AI research. The principle that seemingly "impractical" inquiries like elliptic geometry might one day have a concrete use in the physical sciences begs the question, how to distinguish between the fanciful and the practical if we don't have access to future knowledge? The solution Joseph brought forth was to keep trying at wrong ideas until a correct idea succeeds, much like Edison and the lightbulb. We also spoke at length about creativity and whether or not AI can model the lateral thinking required for leaps of thought like that of Lobachevsky and Bolyai in the late 1800s, and the intricacies of the relationship between the analysis AI models can produce and the ingenuity of a human researcher. We concluded that while AI is limited by the data it is provided by humans, it has the benefit of pattern recognition that might escape the biases of a human researcher. Nevertheless, as was said during Taylor's presentation on machine learning, much of AI learning processes are shaped by the data we provide, which could introduce assumptions into the system based on our decisions. The relationship between machine learning and human intelligence is complex, and requires tradeoffs between many different factors.

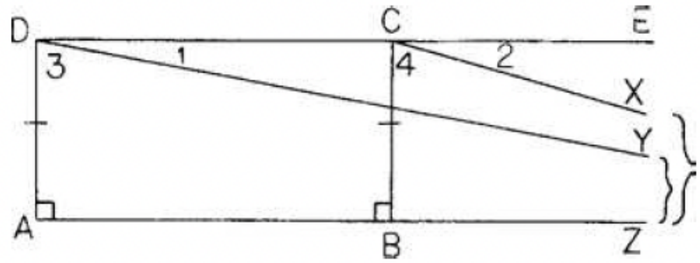
Appendix A - Derivation of Angle-Sum Theorems in Hyperbolic Geometry (Proofs and figures taken from Trudeau, pp. 135-136, 210-212)



(Trudeau, 135)

Part 1: The sum of the interior angles of any triangle in a hyperbolic geometry is less than 180.

1. Let "ABC" be any triangle, as per the hypothesis.
2. Bisect AB at "D" and AC at "E."
3. Draw a line through DE.
4. Draw lines B"F" and D"G" perpendicular to DE.
    a. We now possess what's known as a Saccheri Quadrilateral, a quadrilateral with two right angles at its base, which will be our tool for proving the angle-sum theorem.
5. Draw A"H" perpendicular to the line through DE.
6. Triangles BFD and AHD are congruent, by the A-A-S theorem of congruence.
7. Angle DBF = angle DAH, from the definition of congruence.
8. Angle DBF + angle ABC = angle DAH + angle ABC, by common notion.
9. Triangles CGE and AHE are congruent, by the A-A-S theorem of congruence.
10. Angle ECG = angle EAH, from the definition of congruence
11. Angle ECG + angle ACB = angle EAH + angle ACB, by common notion.
12. Angle FBC + angle GCB = angle BAC + angle ABC + angle ACB, by common notion.
    a. We can now show that the given angle-sums of our Saccheri Quadrilateral are less than 180, completing the proof. Until now, the proof required no discussion of Postulate 5 or its negation, so points 1-12 hold true in any geometry, Euclidean or otherwise.



(Trudeau, 135)

Part 1a. The upper angles of a Saccheri Quadrilateral sum to less than 180 under hyperbolic geometry.

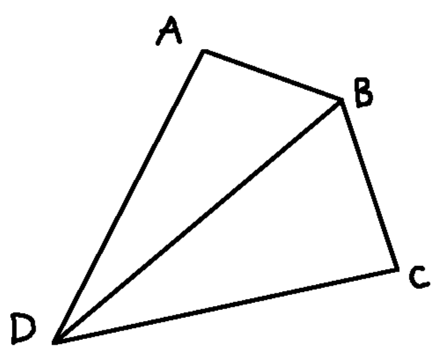
1. Let ABCD be a Saccheri Quadrilateral with a summit CD, as per the hypothesis.
2. Extend CD to "E" and AB to "Z."
3. Through D and C draw D"Y" and C"X" asymptotically parallel to ABZ in the direction of Z.
    a. This requires a deeper explanation of the hyperbolic geometry, but for the sake of brevity I'll explain this as a result coming from the "multiple lines" being parallel to any given line - in this case, ABZ - through a point not on that line - in this case, D and C - that are possible in hyperbolic geometry.
4. DCE is divergently parallel to ABZ
    a. Similarly, this is a consequence of hyperbolic geometry - asymptotic parallel lines partition the space into regions where many lines are parallel to a line through a point.
5. In particular, then, DCE is divergently parallel to ABZ through D.
6. But DY is asymptotically parallel to ABZ through D.
7. Thus, DY enters angle EDA.
8. Similarly, enters ECB.
9. CX and DY are asymptotically parallel to each other in the same direction as they are asymptotically parallel to ABZ.
10. XCDY is a biangle
    a. This is a hyperbolic figure made of two asymptotically parallel lines and a connecting segment unique to hyperbolic geometry
11. Angle CDY is less than angle ECX

a. Not proven here, requires more knowledge of hyperbolic geometry - it is similar to the parallel interior angle theorem of Euclidean geometry

12. YDAZ and XCBZ are biangles
13. With angle DAZ = angle CBZ and DA = CB, as per the definition of of Saccheri quadrilateral
14. So angle YDA = angle XCB.
15. Therefore angle CDA < angle ECB.
16. But angle CDA = angle DCB,
17. So angle DCB < angle ECB.
18. Since DCB + angle ECB = 180,
19. Angle DCB < 90 and
20. Angle CDA < 90.
21. Therefore, the sum of the upper angles of a Saccheri Quadrilateral sum to less than 180 under hyperbolic geometry.

Continuation of Part 1

13. Since angle FBC + angle GCB < 180,
14. Angle BAC + angle ABC + angle ACB < 180.



Part 2. The sum of the interior angles of any quadrilateral in a hyperbolic geometry is less than 360.

1. Let ABCD be any quadrilateral, as per the hypothesis.
2. Construct BD.
3. Then ABD and BCD are triangles.
4. By part 1, angle ABD + angle BDA + angle DAB < 180.

5. Also by part 1, angle DBC + angle BCD + angle CDB < 180.
6. Therefore, angle ABC + angle BCD + angle CDA + angle DAB < 360.

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