Markov Chains and Applications in Finance

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Transition Probability

Mouse in a Maze.

The mouse can only go to adjacent rooms; its decisions are independent of any history and only depend on the current room the mouse is in; the mouse cannot stay in the same room forever.

Figure 1: The Maze
Transition Probability

Let $P_{ji}$ be the probability of going to room $j$ when the mouse is in room $i$ by one move.

- $P_{21} = \frac{1}{3}$ is the probability of moving to room 2 from room 1.
- $P_{11} = 0$ is the probability of staying at room 1 forever.
Matrix Representations

- To make it more computationally efficient: use matrix representations

Transition Matrix: \( P = \begin{bmatrix} P_{11} & P_{12} & \ldots & P_{1r} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ P_{r1} & P_{r2} & \ldots & P_{rr} \end{bmatrix} \)

- To find the transition probability from state \( i \) to state \( j \): look up \( P_{ji} \) in the matrix.
What is the probability of getting to room 5 after 1 step? The starting state matters.

If we flip a coin to determine the starting point of the mouse: say, we put the mouse in room 1 if we get a head, and put the mouse in room 2 if we get a tail.

\[
P(\text{starting from room 1}) = P(\text{starting from room 2}) = \frac{1}{2}
\]

\[
P(\text{ending up in room 5}) = \frac{1}{2} P_{51} + \frac{1}{2} P_{52}
\]
Define every room as a state. What if the probabilities of initial states are different? Say, roll an unfair 5-sided dice to determine the starting room.

\[ S = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} \]

represents the starting state of the mouse.

\[ S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

means that the mouse is in room 1 at the beginning in the fair dice case.
Combining the transition matrix and the distribution of probabilities

1. The distribution of transition probabilities on the states after 1 step starting from state $s$ is the matrix-vector product

$$(Pq)_i = \sum_{j=1}^{r} P_{ij} q_j = P_{i1} q_1 + P_{i2} q_2 + \ldots + P_{ir} q_r$$

2. The probability of ending up in state $j$ after 2 steps starting from state $i$ is:

$$(P^2)_{ji} = \sum_{k=1}^{r} P_{jk} P_{ki}$$
Markov Chains

- In English,
  \[ \text{Next State} = [\text{Matrix of Trans. Probabilities}][\text{Current State}] \]

- The predicted value is based solely on the current value
Applications of Markov Chains

- Predicting Stock Market Trends
  A hypothetical market with trends shown as below:

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Bull</th>
<th>Bear</th>
<th>Stagnant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bull</td>
<td>Bull</td>
<td>0.9</td>
<td>0.075</td>
<td>0.025</td>
</tr>
<tr>
<td>Bear</td>
<td>Bull</td>
<td>0.15</td>
<td>0.8</td>
<td>0.05</td>
</tr>
<tr>
<td>Stagnant</td>
<td>Bull</td>
<td>0.25</td>
<td>0.25</td>
<td>0.5</td>
</tr>
</tbody>
</table>

- For example, this means that the probability of going from the bull market to bear market is 0.075, but the probability of going from bear market to bull market is 0.15.
In the previous example of mouse and maze, a state is defined as the room a mouse is in. In this problem a state is a time period. Assume in this problem a state is one week long.

If we set the current week as bearish, then the vector of the initial state is \[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}.
\]

We can now calculate the probabilities of a bull, bear or a stagnant week from any number of weeks into the future.

1 week from now:

\[
S_1 = \begin{bmatrix}
0.9 & 0.075 & 0.025 \\
0.15 & 0.8 & 0.05 \\
0.25 & 0.25 & 0.5
\end{bmatrix} \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
0.15 \\
0.8 \\
0.05
\end{bmatrix}
\]

5 weeks from now:

\[
\begin{bmatrix}
0.9 & 0.075 & 0.025 \\
0.15 & 0.8 & 0.05 \\
0.25 & 0.25 & 0.5
\end{bmatrix}^5 \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
0.48 \\
0.45 \\
0.07
\end{bmatrix}
\]
52 weeks from now:

$$\begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{52} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.63 \\ 0.31 \\ 0.05 \end{bmatrix}$$

100 weeks from now:

$$\begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{100} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.63 \\ 0.31 \\ 0.05 \end{bmatrix}$$

As the number of weeks go to infinity, the probabilities will converge to a steady state.
Hidden Markov Model (HMM)

Hidden Markov Model for Stock Trading (Nguyen, 2017)

- This is a simplified case because we know the matrix of transition probabilities. But in many cases, we only know stock prices, but we don't know how the market will change.

- Two stochastic processes involved:
  Observable: Stock Prices;
  Unobservable: 'State of the system'

- Basic elements of a hidden Markov model:
  Length of observation data $T$;
  Number of states $N$;
  Observation sequence $O = \{O_t, t = 1, 2... T\}$;
  Hidden state sequence $Q = \{q_t, t = 1, 2... T\}$;
  Possible values of each state $\{S_i, i = 1, 2... N\}$;
  Transition matrix $A = (a_{ij}), a_{ij} = P(q_t = S_j | q_{t-1} = S_{j-1})$;
  Vector of initial probability of being in state $S_i$ at time $t = 1$: $p = (p_i), p_i = P(q_1 = S_i)$;
  Observation matrix $B$.
Hidden Markov Model (HMM)
Hidden Markov Model for Stock Trading (Nguyen, 2017)

- Data:
  SP 500 monthly prices from January 1950 to November 2016

- Observable

- Steps to predict stock prices using HMM:
  1. Choose a fixed time period $T$, calibrate parameters $A, B, p$, decide the number of states $N$;
  2. Move the given data (the given observation sequence) backward to get a new dataset $O^{\text{new}} = \{O_2, O_3...O_{t-1}\}$; compute the vector $p = (p_i)$;
  3. Keep moving backward until we find a new sequence $O^*$, where $p^*_i = P(O^*) \approx P(O)$;
  4. Predict the price at time $T + 1$ using the formula
     
     $O_{T+1} = O_T + (O_{T^*+1} - O_{T^*}) \cdot (P(O \mid A, B) - P(O^* \mid A, B))$
Hidden Markov Model (HMM)

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Results

Figure A1. S&P 500’s predicted prices using the four-state HMM for 40-month out-of-sample period (left) and 60-month out-of-sample period (right).

Figure A2. S&P 500’s predicted prices using the four-state HMM for 80-month out-of-sample period.