Color Space

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Introduction

When we wake up in the morning, our eyes are introduced to the countless colors around us. Colors can help us perceive information, recognize environment around us, create products or even mathematical models. In this paper, we explore a mathematical concept of colors – the color space. We start with the history of the Color Space, followed by its definition. We will then examine the capability of various color spaces that are implemented in different fields, as well as the encoding and transformation of various colors across different color spaces. Finally, we will go through the application of the color space in complex analysis and try bringing connections to other mathematical theories and concepts.

History and Development

The artists today are able to make one color by mixing different other pigments. However, the artists from ancient times often seek for colors that can be directly extracted from a natural source. The investigation of color space came from the potential difference of these two kinds of colors: does human perceive the preexisting colors the same way as mixed colors?

In 1802, British polymath Thomas Young provided us the answer. The colors that human receives are processed through three kinds of photoreceptor cells: red, blue and green. The nerve systems will sense the excitement level of each individual kind of cell to determine the colors our brains recognize. As a result, as long as two colors (one preexisting and one mixed)stimulate the same level of excitement of our photoreceptor cells, we cannot possibly tell them apart. This finding allows us to quantify every color into the the excitement level of each kind of photoreceptor cells.

Around 1853, German polymath Hermann Grassmann, who was also one of the inventors of linear algebra, proposed the idea of constructing colors in a vector space, ultimately led to the concept of color spaces today.

Definition and Terminology

The color space is defined to be an organization of colors which we can use to specify, create and visualize according to our needs. It is usually a vector space that consists of three or more parameters in order to reflect colors under specific applications. These parameters include but are not limited to the following[1]:

- Specific spectrum level of input:
- Hue
- Lightness
- Chroma
- Saturation

Color space Variations

As mentioned above, most of the color spaces we discuss will have their parameters depend on the device of which they are applied to. The differences in parameters result in various color spaces that have different properties and capacities. Some examples of the most commonly used color spaces are[3]:

- **RGB** (Red Green Blue) Color space: mostly used for monitor displays since screens emit light that can be directly received by photoreceptor cells.
- **CMYK** (Cyan Magenta Yellow Black) Color space: mostly used for printers since paints absorb lights instead of emitting them. Thus subtractive coloring with CMY can provide easier manipulation of color combinations. We include black in this case for better grey scale printing quality. (note that RGB color space does not need a black/white parameter for grey scale since light emission is much more precise than paint mixing. Also, unlike paint mixing, the display quality will not change when we combine more than one light).
- ***L*a*b*** Color space: mostly used in color related research. It is able to provide color coordinates that exist outside of the RGB color gamut, resulting in a larger and more complex color space for us to explore. (*L* represents the luminance or brightness of the image, *a* represents the amount of red or green tones in the image whereas *b* represents the amount of yellow or blue tones in the image).
- **HSB** (Hue Saturation and Brightness) Color space: a collection of color spaces that are constructed in order to make the color selection process more intuitive by allowing users to select specific hue/saturation/brightness directly. Similar color spaces include HIS(Hue Saturation and Intensity), HSV(Hue Saturation and Value), HSL(Hue Saturation and Lightness), HCL(Hue Chroma and Lightness) etc. Most of these color spaces are made with linear transformations from the RGB color space.

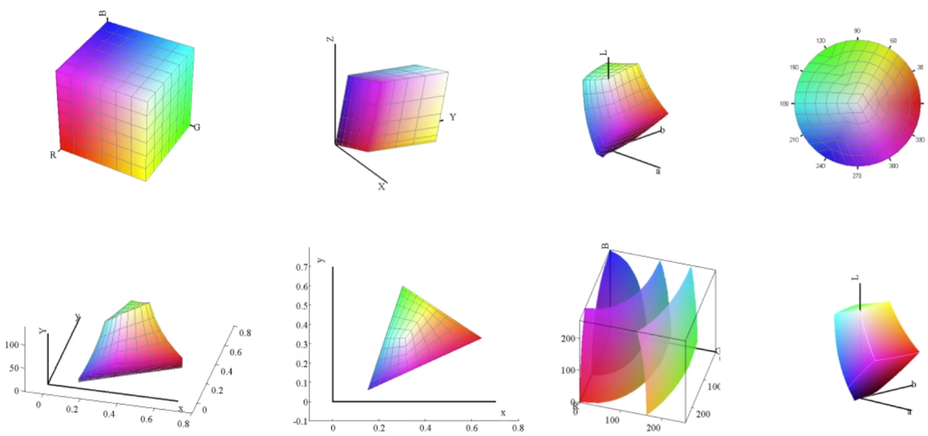

Figure 1. Example of Different Color spaces[2]

Color space Transformations

Since the color spaces are vector spaces, we are able to implement linear or non-linear transformations to convert between different parameters and color spaces. One example would be the transformation between the original linear RGB color space and the modern standard RGB color space, also known as the **sRGB** color space. The sRGB color space is able to provide a more consistent result while adapting to different devices. Consequently, it is a the most commonly used variation of the RGB color space in today's technologies and industries.

In this example, the transformation function we use will be a non-linear function called the *gamma correction* which encode luminance in the sRGB color space.
Suppose that u represents one of the R, G, B color values, we have the parametric curve below to perform the transformation[2]:

$$\begin{aligned} f(u) &= -f(-u), \quad && u < 0 \\ f(u) &= c \cdot u, \quad && 0 \le u < d \\ f(u) &= a \cdot u^{Y} + b, \quad && u \ge d, \end{aligned}$$

With parameters:

$$\begin{gathered} a = 1.055 \\ b = -0.055 \\ c = 12.92 \\ d = 0.0031308 \\ Y = 1/2.4 \end{gathered}$$

Color Encoding

In modern days, artists are able to spot a specific color uniquely using the color space introduced above. But mathematicians are still looking to find a way to represent a specific color not only through a 3D space, but with one line of information using color encoding.

The color encoding that is embedded in most softwares today originated from the RGB color space. In the RGB color space, each vector is corresponding to one of the three RGB colors with a magnitude representing the level of input(luminance) of each color. The standard color encoding is made with uint8 arrays as a base, where the RGB magnitude ranges from $[0, 255]$. Taking the magnitude of each RGB vector from a specific color, we are able to calculate the encoding by converting each magnitude into hexa-decimal numeration, and merge them together in the form of #RRGGBB. The reason behind this method is because 255, which is the maximum magnitude the RGB vectors, happens to be the largest two digit number(FF) in hexa-decimal numberation. Therefore, any arbitrary color in a RGB color space can be represented using only six digits.

Connection with Complex Analysis

In order for us to explore the connection between the color spaces and complex analysis, we need to first recognize the problem that mathematicians encounter when they try to build a geometric representation of complex functions.

In complex analysis, in order to find a geometric representation of a complex number, we need to construct a **complex plane:** it is established by a **real axis** and a perpendicular **imaginary axis** [4]. The real portion of the complex number will be represented on the real axis, and the imaginary part of it will be shown on the imaginary axis. Thus we are able to depict a complex number graphically using a two dimensional plane.

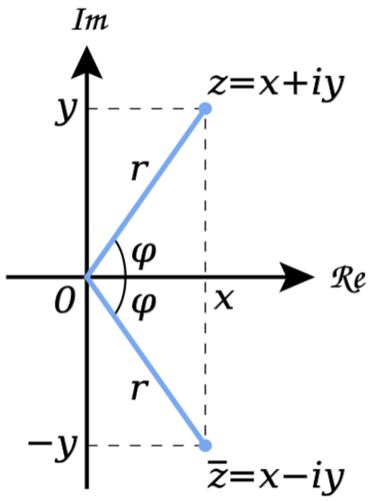

Figure 2. Example of a complex plane [4]

However, when mathematicians attempt to produce a similar graph on complex functions, the problem arises. Unlike a regular function where we use one axis to represent the independent variable and the other one to represent the dependent variable, a complex function requires two axes for each complex variable. When both the independent and dependent variable of the function take on complex values, we are forced to construct the geometric representation in a four-dimensional space, which is very hard to visualize. An example of such a function would be:

$$f(x) = x^2, \qquad x \in \mathbb{C}$$

$$x = 2 + i$$
$$\Rightarrow\ f(x) = 3 + 2i$$

where both the domain and range of the function take on complex values.

To properly produce graphs of complex functions on two-dimension planes, mathematicians used a method called **domain coloring**. This techinique assigns a color to each individual point of the complex plane to visualize the complex function. By assigning points of the complex plane to different colors and brightness, domain coloring allows complex functions to be easily visualized and understood in two-dimensional planes.

The assignment of color in domain coloring is called the **color function**. It is usually a function that maps specific parameter of the complex function onto a color space, allowing us to use the corresponding color to represent the point on the complex plane. One common practice is to represent the complex argument with a hue following the specific color space, and the magnitude of the complex number by other means including brightness and saturation[5].

One simple example of the color function would be the mapping from a HLS(Hue Lightness Saturation) color space[5]:

$$H = \arg z,$$
$$L = \ell(|z|)$$
$$S = 100\%$$

If we apply this simple color function to an example complex function $z^3 - 1$ where $z \in \mathbb{C}$, we will get the result in the following figure:

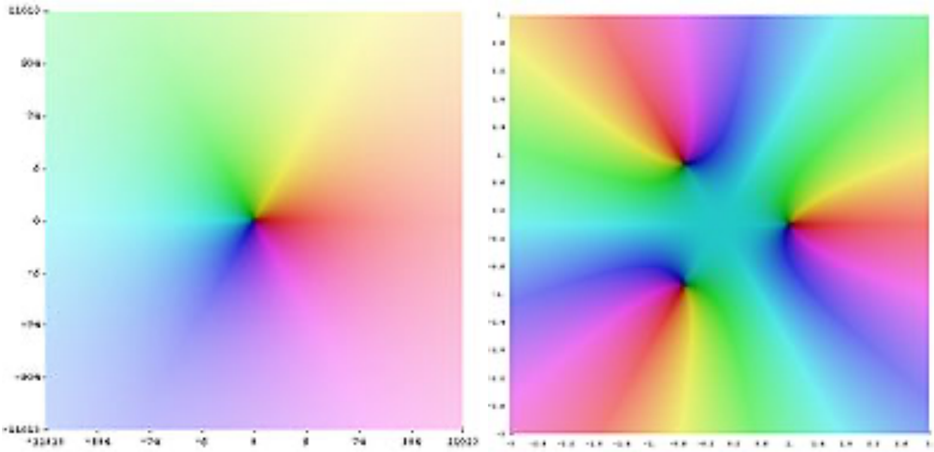

Figure 3. Color function(left) and Domain Coloring result(right) [5]

Domain coloring is a powerful tool that is supported by the concept of color spaces. With the help of domain coloring, we are able to make the research in complex analysis much more efficient. Domain coloring is also a common technique in the study of fractal geometry. One example would be the Mandelbrot set: a set of complex number c for which the function $f_c(z) = z^2 + c$ does not diverge when iterated from $z = 0$ [6].

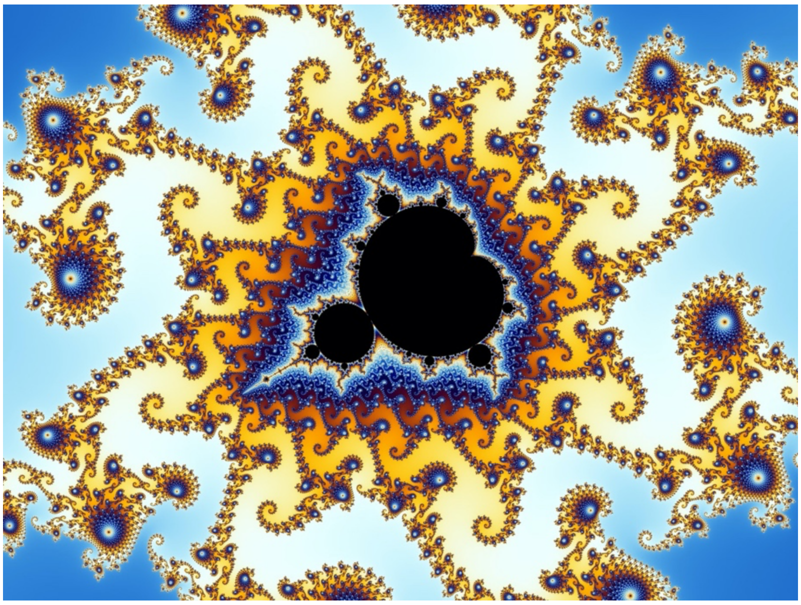

Figure 4. The Mandelbrot set [6]

Conclusion

Mathematics have helped us construct the color spaces, and the color spaces we built also serve as components in the development of new mathematical theories and techinques. Color spaces are the bases of almost any techinology products that relate to display and printing in the modern days. They provide us an overview and a perspective of how colors can be perceived, as well as forming connections all the way from linear algebra to complex analysis and beyond.

References

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