The Mathematics Behind Sudoku

Laura Olliverrie

Based off research by Bertram Felgenhauer, Ed Russel and Frazer Jarvis

Abstract

I will explore the research done by Bertram Felgenhauer, Ed Russel and Frazer Jarvis on the mathematics behind sudoku. Specifically, I dive into their method on enumerating the number of distinct Sudoku grids, the use of computer algorithms to solve puzzles, and the application of abstract algebra and graph theory to enumerate "essentially different" sudoku grids.

Introduction to Sudoku

Sudoku is a logic-based number placement-puzzle typically presented on a 9x9 grid (but there are many variations). Numbered puzzles appeared in French newspapers as early as the late 19th century. Howard Garns is credited with the invention of Sudoku, which was originally named *Number Place*. The first puzzle appeared in the magazine *Dell Pencil Puzzles and Word Games* in 1979. Sudoku was introduced in Japan by Nikoli in the 1980s as *Sūji wa dokushin ni kagiru* and later abbreviated to sudoku which means single number (and is now a registered trademark of a Japanese puzzle publishing company). It first appeared in a US newspaper and *The Times* in 2004 due in large part to Wayne Gould who created a computer program to rapidly produce distinct puzzles.

Here is the puzzle. Good luck!

			4		7		9	
		8	1	5				2
	5			6		7		3
			5				7	6
3	7		9	4	6		5	1
1	6				3			
5		7		2			8	
2				9	5	1		
	8		6		1			

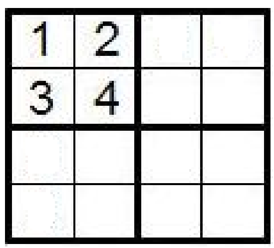
One Rule

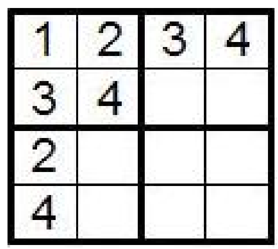
To complete a sudoku puzzle, players insert the numbers 1,2,3,4,5,6,7,8,9 into a 9x9 grid (81 **cells**) subdivided into nine 3x3 **blocks** in such a way that each column, row, and 3x3 block has exactly one occurrence of each number. A certain amount of numbers is provided (known as **givens**) on the grid for each player to aid in starting the puzzle. There is no basic arithmetic required in completely a puzzle, but logical deduction can prove to be very useful in solving the puzzle.

Sudoku is a fun puzzle game that gained widespread recognition throughout the world. Since the introduction of sudoku in a magazine, anyone can now have a new sudoku puzzle to play at the snap of a finger. Sudoku puzzles can be found in newspapers, puzzle-based books, and online sudoku games. With so many puzzles readily available for mass consumption, I wanted to investigate the mathematics behind two questions: (1) Approximately how many distinct sudoku grids are there? And (2) How many sudoku grids are “essentially different”?

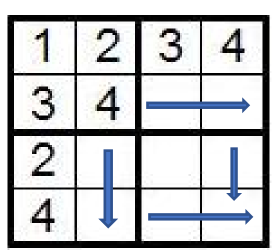
4x4 Case

Let’s begin by exploring the case of a 4x4 grid (Cornell University Department of Mathematics). By learning tricks to enumerate a 4xr grid, we can later use the basis of these ideas to understand Felgenhauer and Jarvis’ work on enumerating the classic 9x9 grid. To count the number of valid 4x4 sudoku grids (applying the One Rule), we must first begin by filling the top left block in standard form, where the first four cells are filled in consecutive order from 1 to 4. We can quickly realize that there 4! ways to relabel the digits in Block 1, thus there are currently 24 distinct grids for a 4x4 grid.





The order in which we can enter 3 & 4 in the first row and 2 & 4 in the first column does not matter since swapping the numbers will preserve the One Rule. Since 1 & 3 occur in the first column of the grid already, 2 & 4 must be placed in the third and fourth cells of that column. Likewise, 3 & 4 must be placed in the third and fourth cells of the first row since 1 & 2 are occupying the first and second cells of row 1. Considering swapping is acceptable, there are 2 ways to order 1 & 2 and 3 & 4 each, so we must account for this in our formula to enumerate the number of distinct cells a 4x4 grid holds. Thus, we have now enumerated 4!*2*2 = 96 distinct grids. We quickly conclude that 4 must be placed in the (3,3)[1] position since the only possibilities are 1 & 4, and 4 occurs in the fourth row, fourth column, and bottom right block already.



[1] To signify the position of a number on the grid, I will use a parenthetical notation to represent the rows and cells in a grid. For example, (1,3) will signify the first row of the grid and the third column (row #, column #).

Now, there are 3 possible entries for the cell in the (4,4) position: 1, 2 and 3. To complete the formula for the 4x4 case, we find that there are 4!*2*2*3 = 288 distinct 4x4 sudoku solutions.

1	2	3	4
3	4		
2		4	
4			1

1	2	3	4
3	4		
2		4	
4			2

1	2	3	4
3	4		
2		4	
4			3

If we are to fill in the three possible grids from above, placing 1, 2, & 3 in the (4,4) position respectively, we find that the third grid is equivalent to the second grid. By reflection across the diagonal from the upper left to the bottom right and relabeling by interchanging 2 and 3, we find a symmetry between the second and third. Thus, there are 2 “essentially different” 4x4 Sudoku grids.

9x9 Case

	B1			B2			B3	
	B4			B5			B6	
	B7			B8			B9	

1	2	3						
4	5	6						
7	8	9						

I will apply Bertram Felgenhauer and Frazer Jarvis method to the count the number of valid 9x9 sudoku grids (following the One Rule). Firstly, I will address the terminology that Felgenhauer and Jarvis use in *Mathematics of Sudoku I*. 3 rows of blocks will be called a **band**, 3 columns of blocks will be called a **stack**, and N will represent the number of distinct sudoku grids.

Once the top left block is filled in standard form, we find there are 9! = 362880 ways of filling B1[2] and N = N_1*9!. Then, Felgenhauer and Jarvus found that we must consider all possible ways to fill in blocks B2, B3, given that B1 is in the standard form above.

1	2	3	{4	6	8}	{5	7	9}
4	5	6	{7	9	a}	{8	b	c}
7	8	9	{5	b	c}	{4	6	a}

Felgenhauer and Jarvis found that there are 20 ways to fill in B1 and B2. A pure top row would consist of B2 and B3 containing cells of {4, 5, 6}|{7, 8, 9} or {7, 8, 9}|{4, 5, 6}, where each block has a top row of numbers in consecutive order. A mixed top row would be the remaining

2 Each B represents each block.

18 ways to fill B2 and B3. Mathematically, there are $(3!)^6$ possible configurations to complete the first band starting with the pure top row (each set of three numbers can be written in 3! = 6 different ways), and $3 \times (3!)^6$ possible configurations to complete the mixed top rows. Once they put together their mathematical findings, they reported that there are $2 \times (3!)^6 + 18 \times 3 \times (3!)^6 = 56 \times (3!)^6 = 2612736$ possible completions to the top three rows. Therefore, the number of possibilities for the top three rows of a Sudoku grid is 9! × 2612736 = 948109639680.

How many distinct sudoku grids are there?

Through lexicographical reduction, permutation reduction, and column reduction, Felgenhauer and Jarvis found that they could reduce the number of possibilies to improve calculations. They, with the aid of work done by Ed Russel, found that there are 44 distinct completions for blocks B1-B3. By setting the variable *N* = number of distinct Sudoku grids and letting the variable *C* = 1 of the 44 possibilities, the enumeration of ways that *C* can be completed to a full Sudoku grid (n_c) and possibilities for B1, B2, and B3 that are equivalent to *C* (m_c) was formalized as $N=\Sigma_c m_c n_c$. Felgenhauer programmed a backtracking algorithm, a depth-first search that incrementally builds candidates to the solutions, and abandons a candidate ("backtracks") as soon as it determines that the candidate cannot possibly be completed to a valid solution. The run tim was considerable low and they found that 6,670,903,752,021,072,936,960 (6.67 x 1021) grids are possible by using logic and brute-force computation.

How many sudoku grids are "essentially different"?

Jarvis and Russel wrote a second paper researching the symmetries involved in Sudoku grids. Distinct solutions were regarded as different even when they could become equivalent through some form of transformation. The symmetry group G of a Sudoku grid consists of all the transformations of the square and more,

1. Relabeling the nine digits.
2. Permuting the three stacks.
3. Permuting the three bands.
4. Permuting the three columns within a stack.
5. Permuting the three rows within a band.
6. Any reflection or rotation (from the list of symmetries of a square).

Russel and Jarvis, in *Mathematics of Sudoku II*, regard two grids as equivalent if one can be transformed into the other by relabeling. If there are no such symmetries to be used, the grids are essentially different. Thus, an enumeration of the number of grids which are fixed up to equivalence by a given symmetry, that is, the grids which are transformed by the symmetry into an equivalent grid must be done. Russel and Jarvis used a program named GAP to work with groups arising by permuting sets. When symmetries such as rotation, reflection, permutation, and relabeling are factored in, there are only 5,472,730,538 essentially different grids.

Computing

There are many forms of computing to compute the symmetries of grids. One such way is brute-force search which is problem-solving technique that consists of systematically enumerating all possible candidates for the solution and checking whether each candidate satisfies the problem's statement. Another way is backtracking which is the algorithm that Felgenhauer used. Large sets of solutions can be discarded without being explicitly enumerated. Often much faster than brute force enumeration. Solving puzzles of $n^2 \times n^2$ grids of $n \times n$ blocks

Conclusion

Felgenhaur, Jarvis, and Russel found utilized many aspects of rudimentary and advanced mathematics such as factorization and abstract algebra to answer two interesting questions on the mathematics behind Sudoku.

References

Bertram Felgenhauer, Frazer Jarvis, *Mathematics of Sudoku I*, submitted to Mathematical Spectrum, http://www.afjarvis.staff.shef.ac.uk/sudoku/felgenhauer_jarvis_spec1.pdf

Ed Russekk, Frazer Jarvis, *Mathematics of Sudoku II*, http://www.afjarvis.staff.shef.ac.uk/sudoku/russell_jarvis_spec2.pdf

Cornell University Department of Mathematics, *The Math Behind Sudoku*, http://pi.math.cornell.edu/~mec/Summer2009/Mahmood/Intro.html