The Math Behind the Music: Deconstructing the nature of harmonics, sound, and the music we make

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1 Introduction - The Universality of Music

The concept of music as it is commonly interpreted is largely experienced only by humans. Other organisms may possess some of the hallmarks of human musical expression, like rhythm or interval recognition, but their combination into musical songs, scores, and expression is limited to the human experience. Why is this the case? Experts are divided on the origin and evolutionary impetus for music existing as an important cultural activity, but it seems to be near-universal among human societies - music is found in every people group around the world [5]. Music's origins may be found in the usefulness of repetition and ritual in communicating between and among social groups, either for hunting, gathering, or participating in culturally significant activities like funeral practices [5]. Wherever these practices arise from, to understand their structure one must analyze the fundamental building blocks that all music makes use of: rhythm and pitch. Building off of these foundations, one can explore how the series of harmonics shapes modern musical patterns.

2 Rhythm and Pitch

The two most fundamental building blocks of music are rhythm and pitch. Both are intimately connected with the medium through which humans experience music: timing and repetition. Rhythm encompasses the overarching patterns of sound and silence that give music its “musical” quality - these larger systems of repetition and sequence are recognized by the brain as more significant than white noise.

On the other hand, pitch is tied to the frequency of the fundamental tones that music is built out of. The timing here is too fine-grained for the ear to register as distinct noises; rather, we perceive pitch as the “highness” and “lowness” of a sound [2]. The higher the frequency of the tone, the higher the pitch. The human ear is able to register frequencies between 20 Hz and 20 kHz on average, though this range is variable based on age and relative hearing damage. The most peculiar aspect of pitch and its perception by humans is the near-universal acceptance that doubling a frequency produces a quality of “sameness” in pitch, to the point that in western music, a doubled frequency is given the same name as the fundamental tone - this is called an “octave,” the significance of which we will soon see [4].

3 Pythagoras and the Perfect Fifth

The octave is merely the first significant ratio humans incorporated into musical vernacular. Perhaps more critical to the development of Western music was the perfect fifth, which according to legend was formulated by Pythagoras in the 6th century BCE. As the story goes, Pythagoras passed by a blacksmith’s forge and, hearing harmonies he perceived as pleasing, entered in to investigate. The results of this observation showed that string lengths with integer relationships to each other produced the most pleasing tones when played together - specifically, the perfect fifth [3]. To produce such a pitch, Pythagoras found that holding a string at half length and equal tension produced frequency 3/2 the original string length frequency [3]. Today, this pitch is known as the perfect fifth. From these foundations, the field of harmonics opens wide.

4 Harmonics

Harmonics, in mathematics, are waves with frequencies that are integer multiples of a root wave, called the “fundamental frequency.” In nature, whenever a note is played, harmonic frequencies tend to arise from resonances in the medium of the wave - whether that be the shape of the coastline that ocean waves roll upon or the chambers of a violin vibrating harmoniously with the strings. Integer ratios represent common intervals, with the 2:1 being called an "octave," a 3:2 ratio called a "perfect 5th," and a 4:3 ratio called a "perfect 4th." [3] These harmonics extend upward, progressively increasing in wavelength, ad infinitum, producing the pitches the human ear finds pleasing. From these ratios the twelve-tone scale is produced, and the foundation of Western music is created [4].

5 Tuning and Scales

The twelve-tone scale begins with the perfect fifth - simply extend upward 6 perfect fifths and downward 6 perfect fifths and divide out the factors of two (octaves) and you will create a system of twelve tones, typically called the chromatic scale. This scale is the basis of almost all western music. To denote the chromatic scale, musicians use seven letters (A through G) and a system of sharps (denoted by the # symbol) and flats (denoted by a lowercase b), arranged according to the following scale:

A A#/Bb B C C#/Db D D#/Eb E F F#/Gb G G#/Ab [2]

With a fundamental frequency determined, this works well - all the tones are pure harmonics. However, there is the issue of transposition - using these ratios, a perfect fifth above a perfect fifth doesn't line up with a second above the octave. Furthermore, the flats and sharps don't line up exactly - G# is not the same as Ab, for example [4].

To solve this problem, different tunings allow for equal space between these semitones. On the 12-tone equal temperament, for example, each semitone is 12th root of 2 times frequency of previous semitone, which means each note on 12-note scale has same distance between them. This allows for easy transposition between keys - an A major scale is identical in ratio to C# major scale [4]. With this flexibility, something must be sacrificed: equal temperament doesn't map exactly onto harmonic systems. In fact, no system using the chromatic scale can have all intervals exactly equal to harmonics! Using the system of cents (1 cent = 1200th root of octave ratio = 100th root of semitone ratio), the 12-tone equal temperament major 3rd is off by 14 cents, which can be detected by a trained ear [4]. Regardless of its shortcomings, 12-tone equal temperament has gained widespread acceptance among Western musicians today.

6 Timbre

But what makes sounds sound different? Why does a chromatic scale played on a guitar sound so distinct from a chromatic scale played on a piano? Surprisingly, the answer can be found in harmonics as well. When an instrument is played, different overtones will dominate the sound, and stronger overtones influence the sound quality [3]. In addition, the larger patterns of how the sound develops over time (called the attack and the decay of the note) affect both the beginning and the end of the sound wave. These factors combined create timbre - the unique sound that is different for every instrument or voice played in music. To illustrate the difference the timbre of a tone can make, consider three kinds of waves - sine, square, and sawtooth waves. Sine waves, the familiar wave shape from pools and vibrating strings, produces a smooth, round timbre, whereas the square wave is much sharper and buzzier. Sawtooth waves are even sharper, and sound choppy. The square and sawtooth waves can be considered aggregations of the sine wave, with different harmonics emphasized in different patterns (see Appendix A for their Fourier analysis) [3]. Different instruments produce different timbres, and provide the rich multiplicity of sounds we enjoy in modern music.

7 Conclusion and Discussion

In conclusion, music is at once extremely rooted in the mathematical concepts of harmonics and wave mechanics, yet organic and uniquely human in its application. What inspired our evolutionary ancestors to consider these ratios appealing? And how can we apply these mathematical principles to create new forms of rhythm and harmony? In our class discussion, we pondered these questions and reflected on how much of music is mathematical and how much of it is organic and instinctual. To be sure, while the roots of music are found in math, its branches extend far beyond the quantifiable into realms of culture, memory, and human nature - which brings to mind the universal applicability of math to these fields. No matter the discipline, mathematical analysis can shed new light on the foundational principles and encourage new perspectives to take root.

A Fourier Analysis of Wave Types

A.1 Sine Wave

Consider the ordinary sine wave:

$$f(x) = \sin(x) \tag{1}$$

Because this wave is periodic with period 2π, using the method of Fourier sine series, we can define this as:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx) \tag{2}$$

and by observation we can determine that

$$A_n = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases} \tag{3}$$

This means that only the fundamental frequency is present in the sine wave.

A.2 Square Wave

Consider the square wave:

$$g(x) = \begin{cases} 1 & \sin(x) > 0 \\ -1 & \sin(x) < 0 \end{cases} \tag{4}$$

Because this wave is periodic with period 2π, using the method of Fourier sine series, we can define this as:

$$g(x) = \sum_{n=1}^{\infty} A_n \sin(nx) \tag{5}$$

By definition, the coefficients A_n can be determined using the formula:

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)dx \tag{6}$$

Simplifying this, we find that

$$A_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{7}$$

So the square wave is formed from the odd-valued harmonics with amplitudes the inverse of the harmonic multiple.

A.3 Sawtooth Wave

Consider the sawtooth wave:

$$h(x) = \left(\frac{x}{\pi} + 1\right)\%(2) - 1 \tag{8}$$

Because this wave is periodic with period 2π, using the method of Fourier sine series, we can define this as:

$$h(x) = \sum_{n=1}^{\infty} A_n \sin(nx) \tag{9}$$

By definition, the coefficients A_n can be determined using the formula:

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin(nx)dx = \frac{2}{\pi^2}\int_{-\pi}^{\pi} x\sin(nx)dx \tag{10}$$

Simplifying this, we find that

$$A_n = \frac{4}{n\pi}(-1)^{n+1} \tag{11}$$

So the sawtooth wave is formed from alternating positive odd harmonics with negative even harmonics, with amplitudes the inverse of the harmonic multiple.

References

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