Turing Patterns

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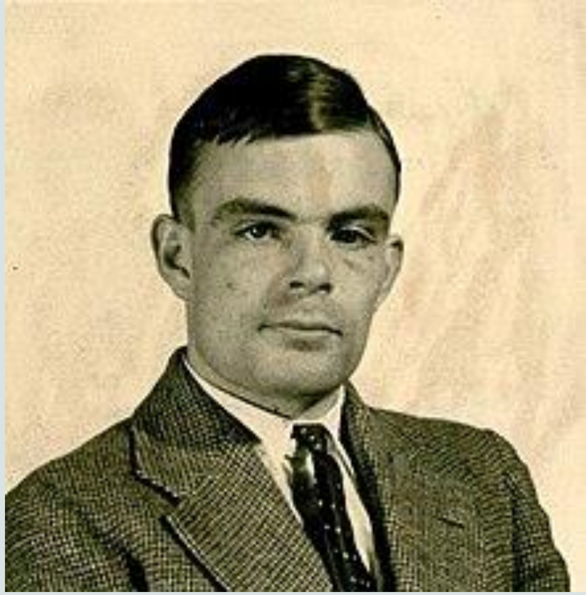
Alan Turing

Who was Alan Turing?

- An English mathematician, computer scientist, logistician, cryptanalyst, philosopher, and theoretical biologist
- Well-known for his role in the creation of the Enigma Machine during WWII and in the early creation of modern-day computers
- Died June 1954 from suicide or accidental cyanide poisoning



Turing and Mathematical Biology

- Published *The Chemical Basis of Morphogenesis* in 1951
- **Morphogenesis**: the development of patterns and shapes in biological organisms

Turing’s Idea

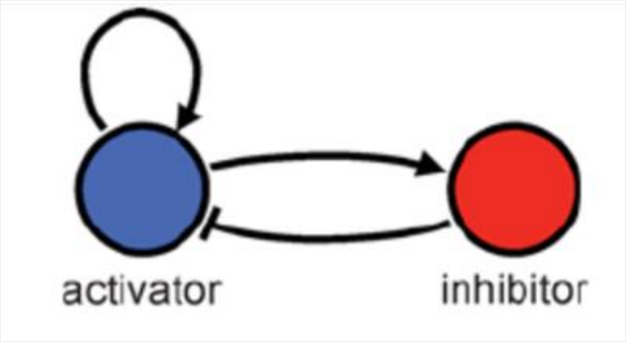
Important Terms

- Morphogen: a signaling molecule that acts directly on cells to produce specific cellular responses depending on its local concentration
  - *Diffuses from a localized source to create a concentration gradient*
- Diffusion-Reaction System: local chemical reactions in which substances are transformed into each other and diffusion causing the substances to spread out over a surface in space
- Perturbations: a disturbance of motion, arrangement, or state of equilibrium

Diffusion-Reaction Systems

- People thought that diffusion created stable conditions (aka one color)
    - Diffusion is a dissipative system
- Turing conjected that diffusion actually destabilizess these chemical systems, creating patterns
- The system starts with 2 chemicals: an activator and an inhibitor

Diffusion-Reaction Systems



Activator

- Substance that promotes the production of both itself and the inhibitor
- Creates a positive feedback loop that amplifies small perturbations

Inhibitor

- Substance that suppresses the production of the activator, and therefore itself
- Creates a negative feedback loop that counteracts the effects of the activator
- Plays a critical role in in controlling the spatial extent and stability of the patterns

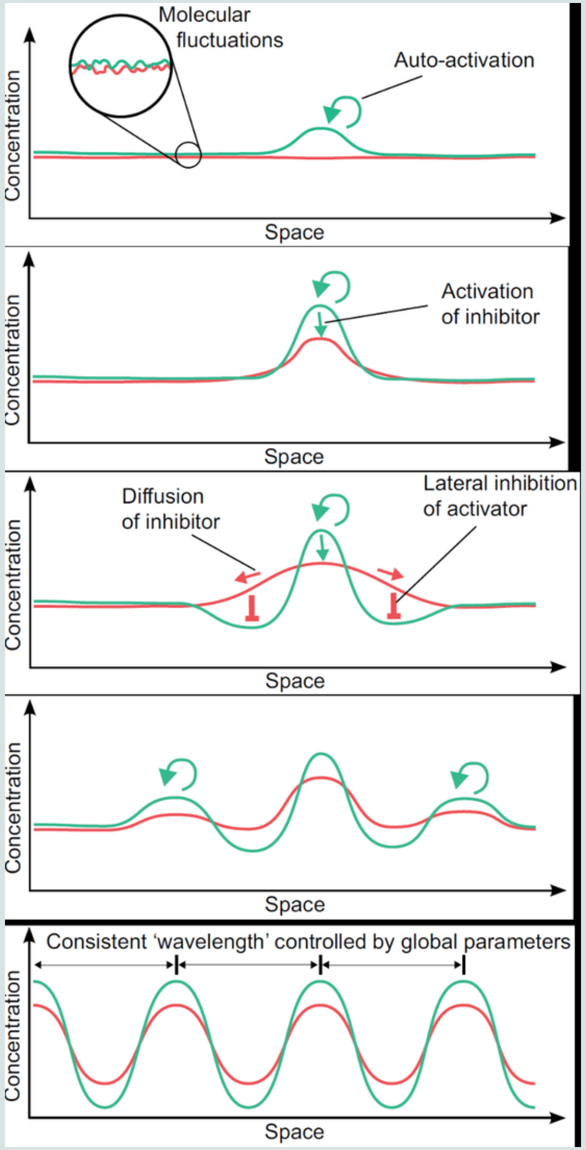
*Important to note that the inhibitor *must* diffuse faster than the activator

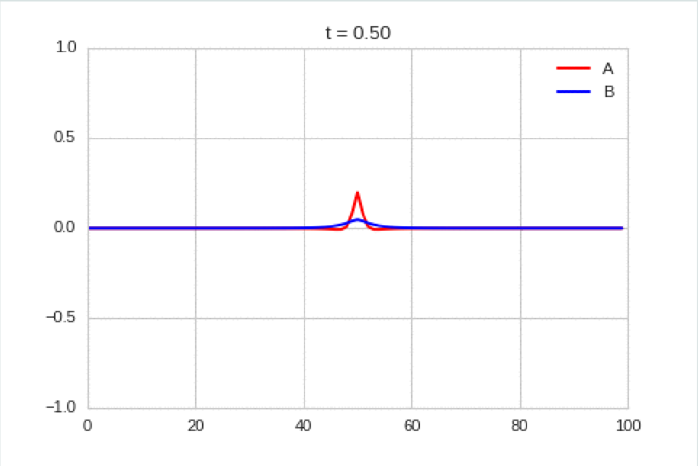
Cheetah-Fire Analogy

- Imagine that you have a cheetah with no spots
- Think of its fur as a dry forest where fires break out in multiple locations
- However, there are also firefighters stationed throughout the forest and they can move faster than the spreading fires
- They must put out the fires from the edges, leaving charred spots throughout the forest

Cheetah-Fire Analogy

- Activator: fire
  - Makes more of itself and brings in more firefighters
- Inhibitor: firefighter
  - Reacting to the fires and extinguishing them
- Both the firefighters and fires diffuse throughout the forest
- Key to putting out all the fires/the cheetah getting its spots is that the inhibitor moves faster than the activator
- By adjusting these variables, we can get many different patterns using Turing's rules


Molecular fluctuations
Auto-activation
Concentration
Space
Activation of inhibitor
Diffusion of inhibitor
Lateral inhibition of activator
Consistent 'wavelength' controlled by global parameters


t = 0.50
A
B
1.0
0.5
0.0
−0.5
−1.0
0
20
40
60
80
100

Examples of Turing Patterns



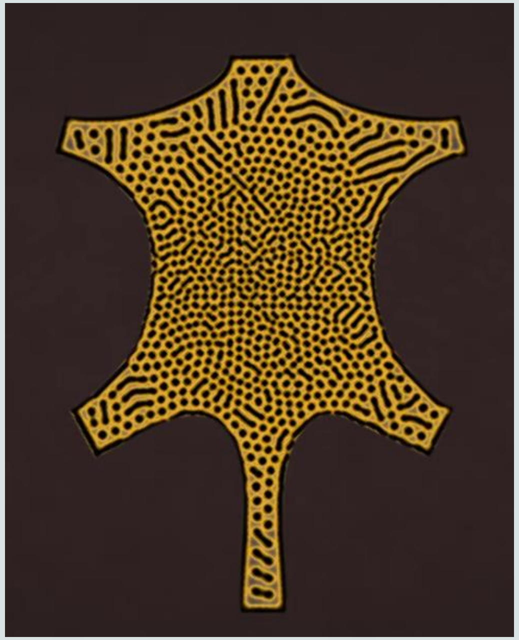


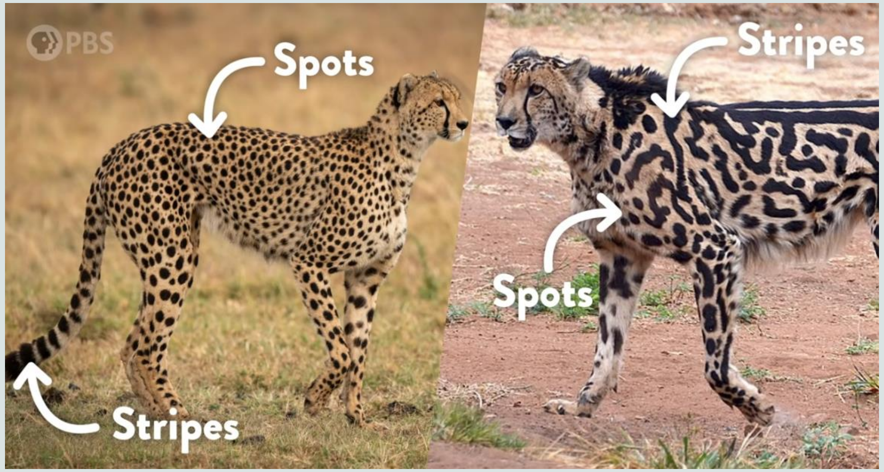

Turing Instability

- Phenomenon in reaction-diffusion systems where homogenous states become unstable and spontaneously give rise to spatial patterns
  - *Homogenous state: when concentrations of activator and inhibitor are uniform*
- This instability arises from the interplay between diffusion and reaction processes within the system depending on their diffusion coefficients and rates of reaction
- As perturbations grow, they eventually reach a size where they become self-sustaining, leading to the formation of spatial patterns
  - *Patterns take on different forms based on system parameters*

Stripes vs Spots

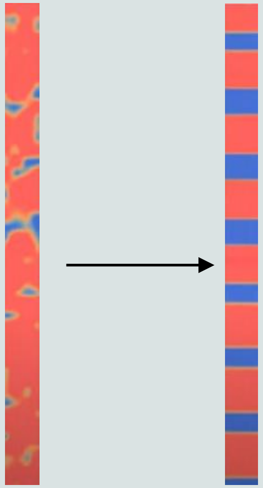
- Spots and stripes are two versions of the same thing
- Depends on parameters in reaction-diffusion equation
- Identical rules playing out on different surfaces
- When Turing patterns play out on irregular surfaces like an animal's body, different patterns can form on different parts

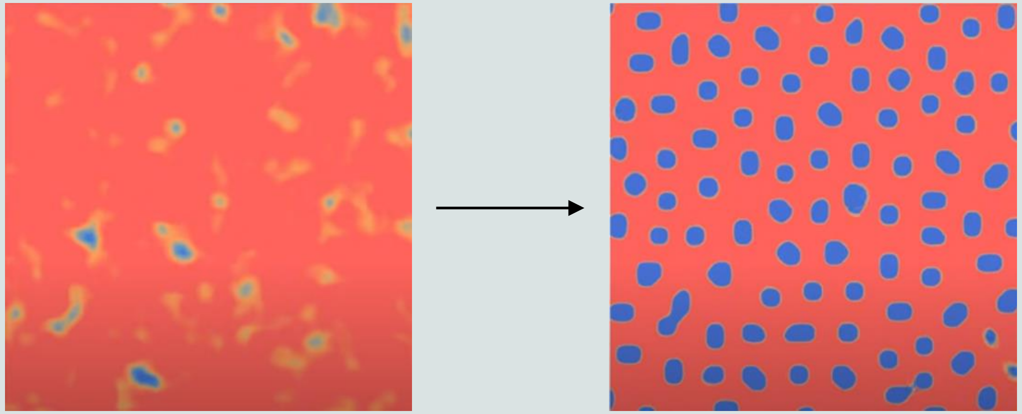




Stripes vs Spots

- Formation of spots vs stripes can also depend on the surface size





Now…
Math!!

Partial Differential Equations

- PDEs are mathematical equations with two or more independent variables, an unknown function dependent on these variables, and partial derivatives of the unknown function with respect to the independent variables
- Computers a function between various partial derivatives of a multivariable function
- Used to aid in the solution of physical problems involving several variables
    - *Reaction and diffusion*

Turing’s PDEs

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + f(u, v)$$
$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + g(u, v)$$

Where:

- u and v are the concentrations of the activator and inhibitor substances
- D_u and D_v are the diffusion coefficients
- ∇^2 is the Laplacian operation, representing the spatial variation or curvature
- $f(u,v)$ and $g(u,v)$ represent the reaction terms, describing how the concentrations of u and v change due to chemical reactions

Changing Parameters

- Diffusion coefficients D_u and D_v
  - *Increasing these can promote the formation of more homogenous patterns with smoother transitions between areas of high and low concentration*
  - *Decreasing these can lead to the formation of more localized patterns with sharper transitions between areas of high and low concentration*
- Reaction rates $f(u,v)$ and $g(u,v)$
  - *Increasing rates of activator production or inhibitor inhibition can lead to amplification of spatial variations and development of more pronounced patterns*
  - *Decreasing these rates can result in suppression of pattern formation or emergence of simpler patterns*

Changing Parameters

- Initial conditions
  - *Alters the starting point from which the system evolves over time*
  - *Varying initial concentrations of the activator and inhibitor can lead to the formation of different patterns or no patterns at all*
  - *Final patterns may exhibit different characteristics depending on initial state of the system*

Different Reaction Terms

- Turing did not include functional forms of reaction terms in his paper
- Instead, he only provided qualitative descriptions of the activator-inhibitor interactions and their effect on pattern formation

Gierer-Meinhardt Pattern Formation

$$\frac{\partial u}{\partial t} = \nabla^2 u + a + \frac{u^2}{v} - bu,$$
$$\frac{\partial v}{\partial t} = D\nabla^2 v + u^2 - cv,$$

Where a, b, c > 0, D > 1

- Changing these parameters can lead to different pattern formations, but this system favors spots
- https://visualpde.com/sim/?preset=GiererMeinhardt

Gierer-Meinhardt Pattern Formation

- By changing the initial conditions, we can observe the instability of stripe patterns

$$u(0,x,y) = 1 + \cos\left(\frac{n\pi x}{L}\right), \quad v(0,x,y) = 1,$$

- Where n is an integer
- https://visualpde.com/sim/?preset=GiererMeinhardtStripeICs

Gierer-Meinhardt Pattern Formation

- A common way to get a stripey/labyrinth pattern using this model is to saturate the self-activation term u^2/v

$$\frac{\partial u}{\partial t} = \nabla^2 u + a + \frac{u^2}{v(1+Ku^2)} - bu,$$
$$\frac{\partial v}{\partial t} = D\nabla^2 v + u^2 - cv,$$

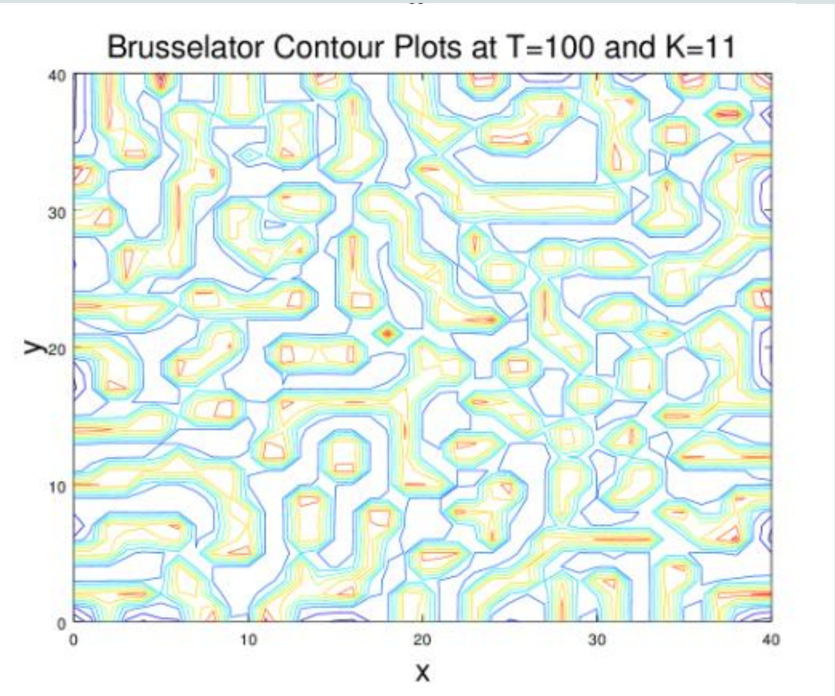
- Where K>0 is a saturation constant
- https://visualpde.com/sim/?preset=GiererMeinhardtStripes

Brusselator Model Showing Turing Instability

- 2 coupled partial differential equations, each representing one species of a two species chemical reaction

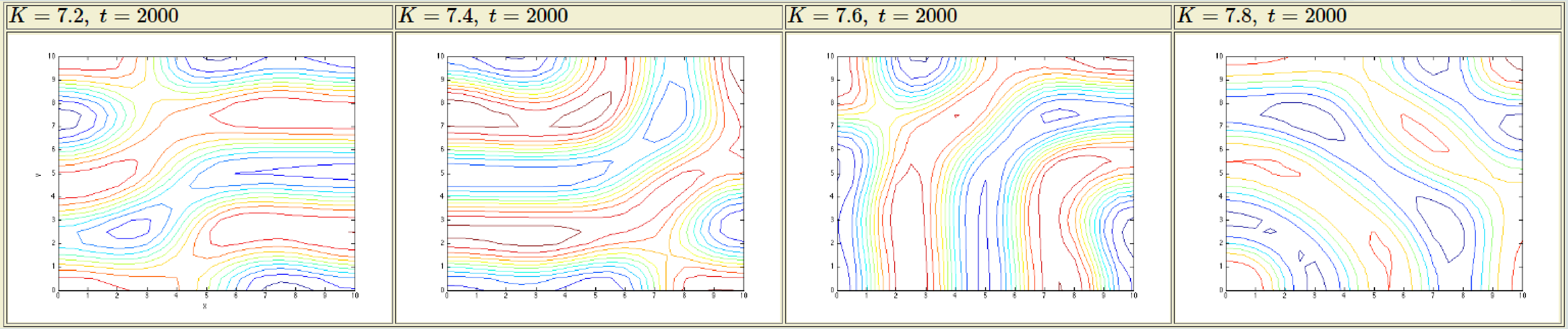
$$p_t = D_p \Delta p + p^2 q + C - (K+1)p$$
$$q_t = D_q \Delta q - p^2 q + Kp$$

$p(x,y,0) = C + 0.1$ for $0 \leq x, y \leq 40$
$q(x,y,0) = \frac{K}{C} + 0.2$
$u_n(x,y,t) = 0$ on rectangle boundary, for all $t \geq 0$



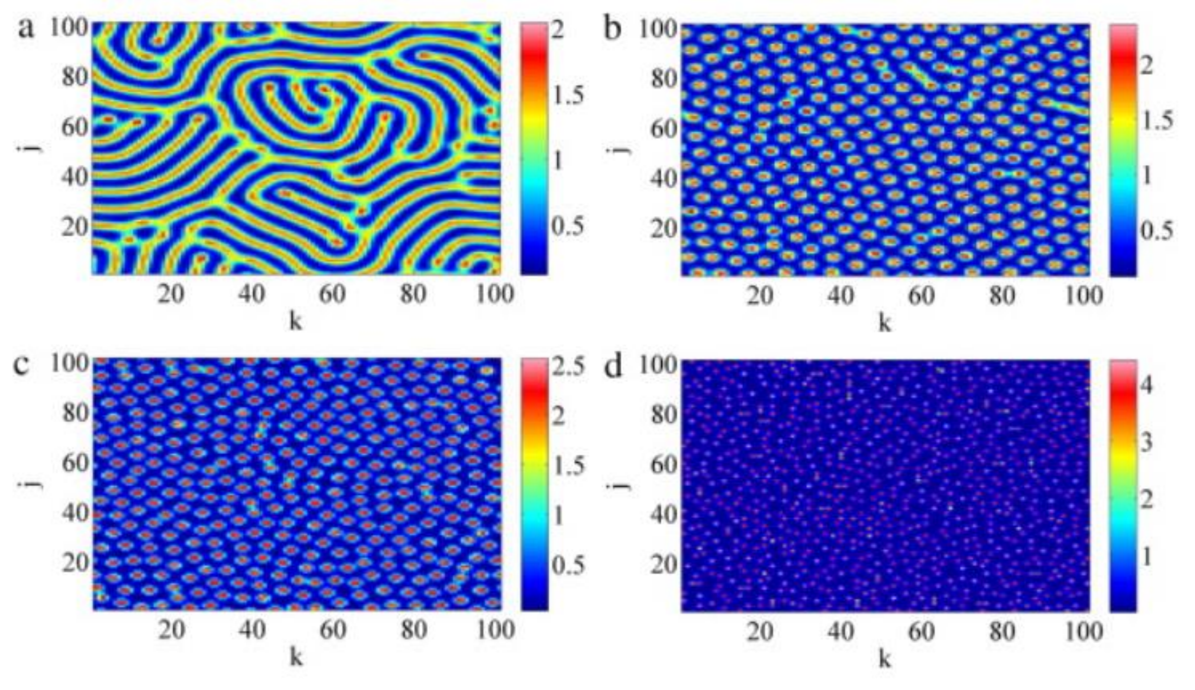
Brusselator Model: Varying K Values

- Wanted to look closer at the K values 7<K<8, t=2000
- Also narrowed the grid to [0,10] x [0,10]



Advancements Since Turing

Advancements After Turing



- 2d spatial pattern of activator concentration at a fixed time
- Activator diffusion coefficient D_u was varied on an interval where system is Turing-unstable; D_v fixed at 0.2

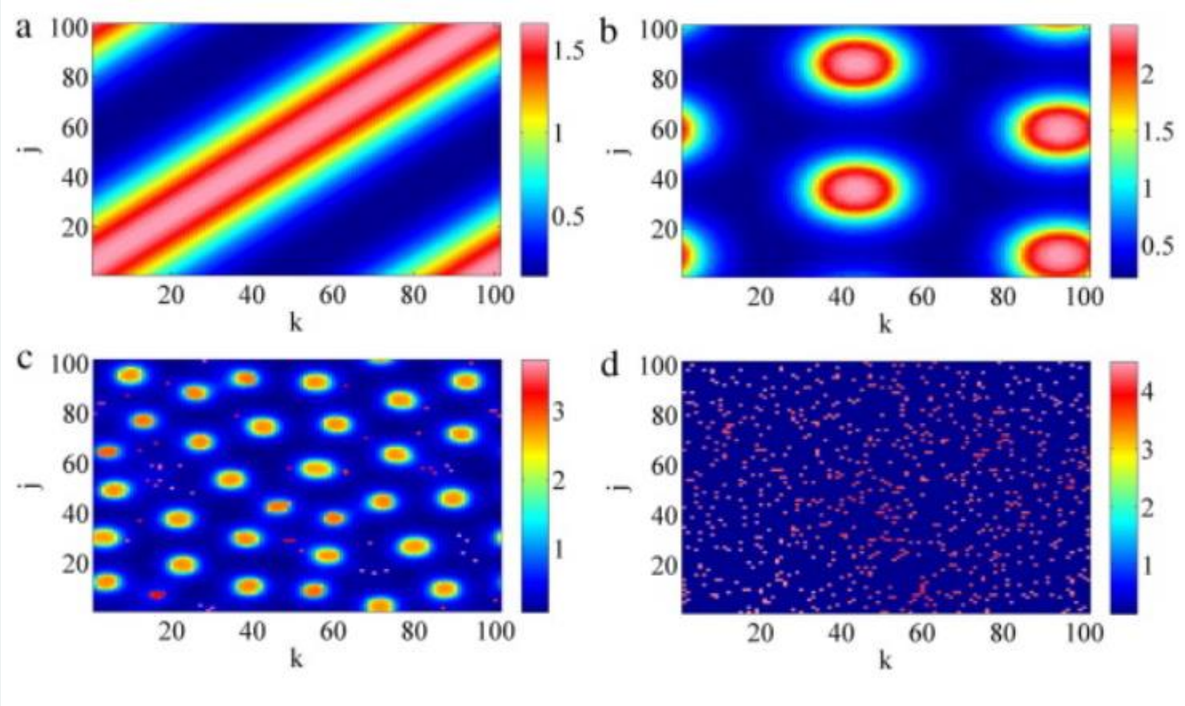
a) D_u = 0.016

b) D_u = 0.014

c) D_u = 0.012

d) D_u = 0.005

Advancements After Turing



- D_v fixed at 0.2

a) $D_u = 0.015$

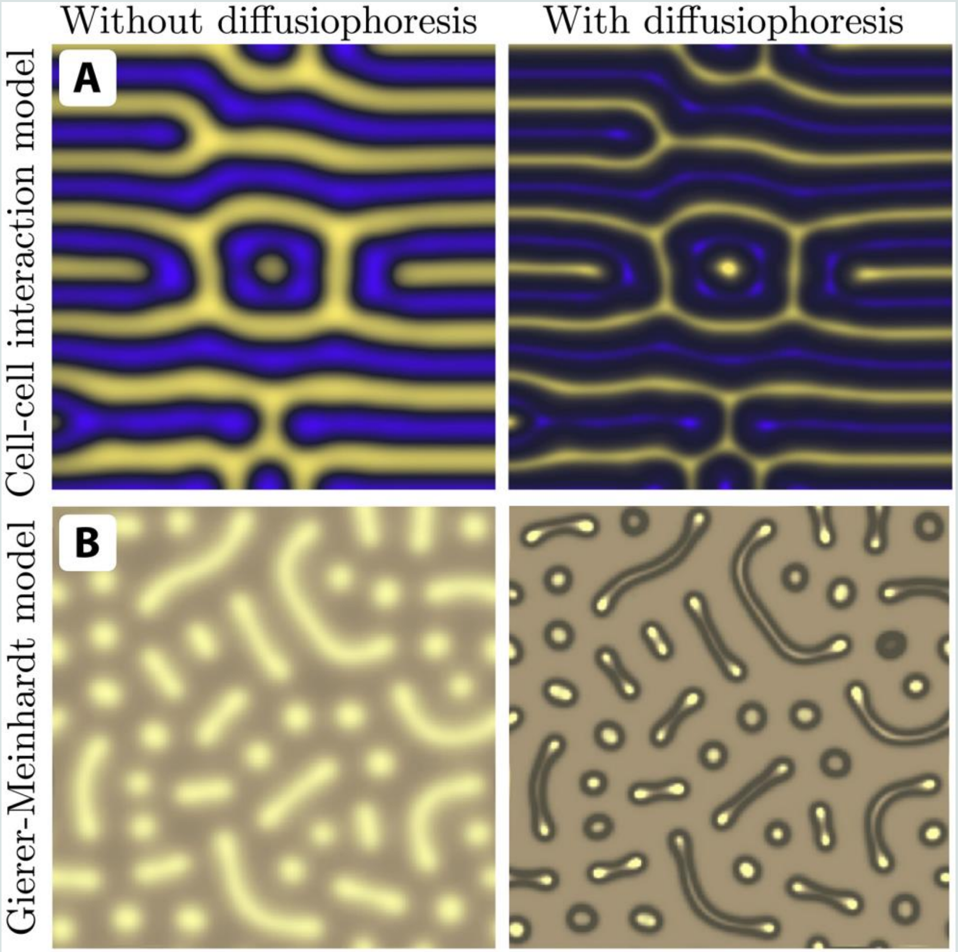
b) $D_u = 0.013$

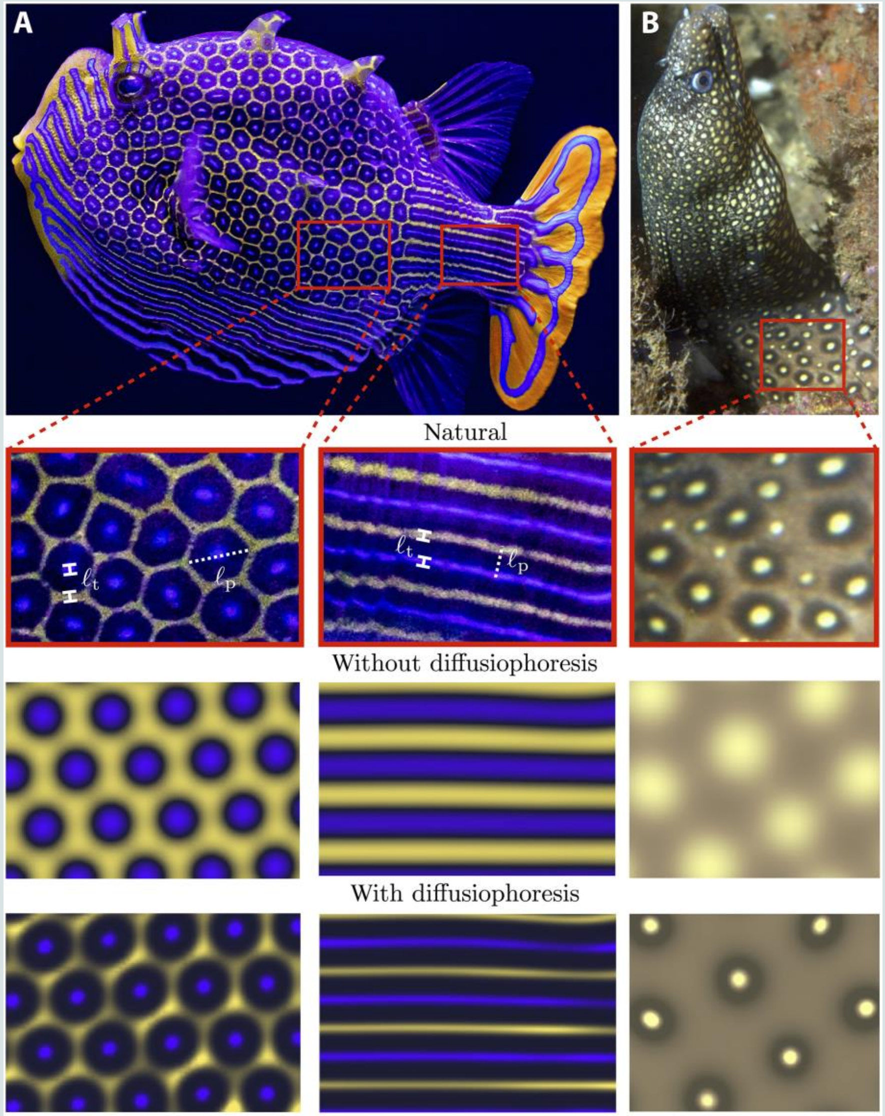
c) $D_u = 0.011$

d) $D_u = 0.005$

Diffusiophoresis

- Turing’s explanation of diffusive transport results in patterns with shallower gradients than we observe in nature
- Diffusiophoresis: the propulsion of colloids by a chemical gradient
- Proposition that diffusiophoresis is involved in Turing patterns, promoting color sharpening like we see in nature
- Chromatophores: cells containing pigment
    - *Respond diffusiophoretically to physiological reactions*


Without diffusiophoresis
With diffusiophoresis
Cell-cell interaction model
Gierer-Meinhardt model
A
B


A
B
Natural
Without diffusiophoresis
With diffusiophoresis

Introducing Computers

- Computers are used for…
  - Numerical simulation: computers solve the reaction-diffusion equations that describe Turing patterns using numerical methods
  - Parameter exploration: researchers can explore the effects of different parameter values systematically
    - i.e. diffusion coefficients, reaction rates, initial conditions on formation, stability, and characteristics on Turing patterns
  - Visualization:
    - Color maps, contour plots, and surface plots help show the concentration of activator and inhibitor substances

Introducing Computers

- Comparison with experimental data: compare simulated Turing patterns with experimental data from chemical, biological, or physical systems
  - *Gain insights into mechanisms underlying pattern formation in real-world systems*
- Parameter estimation and model fitting: iterative process comparing simulated patterns with experimental observations by adjusting model parameters to minimize the difference between simulated and observed patterns

Generative Art

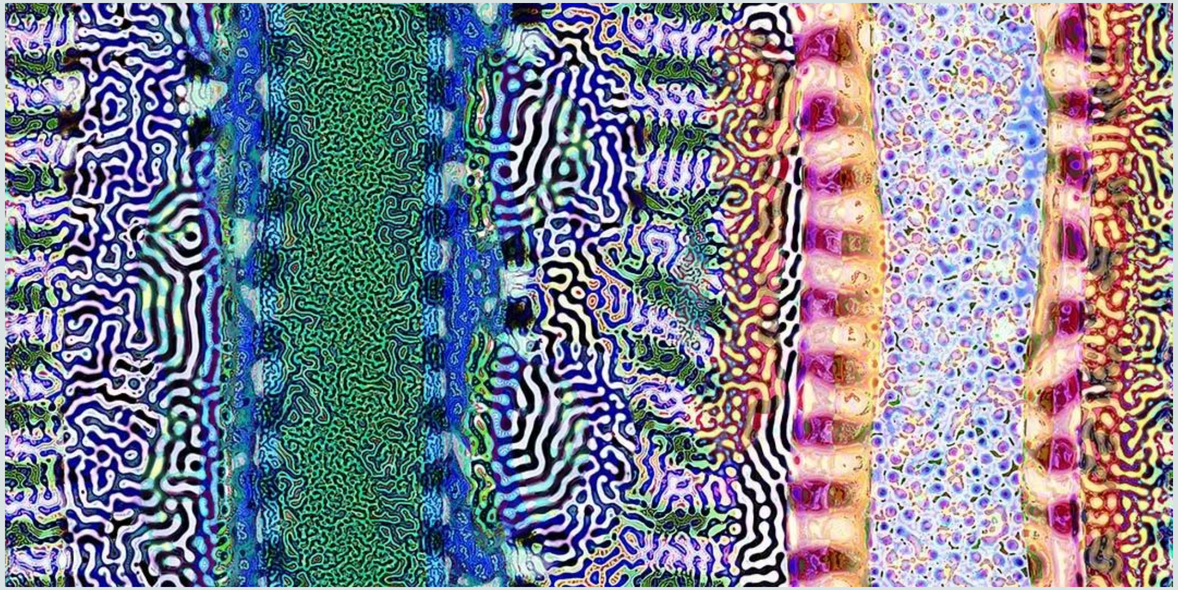
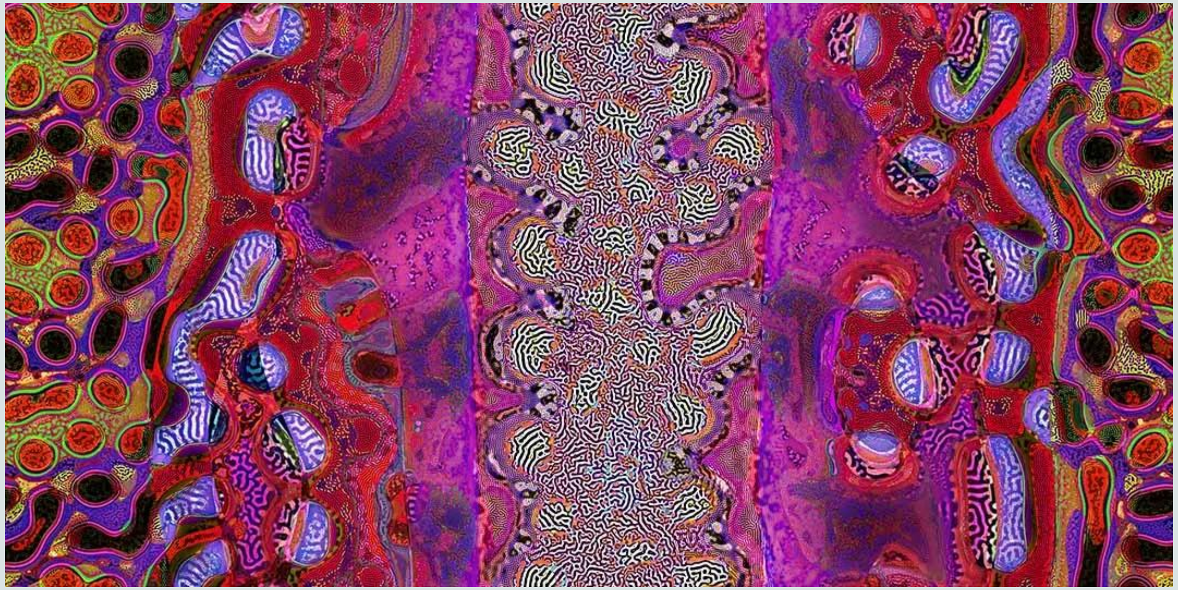
- Generative art: art that in whole or in part has been created with the use of an autonomous system – one that is non-human and can independently determine features that would otherwise require human input
- Made with the help of some outside system (computer program, algorithm)
- The beauty of generative art is that you never know exactly what you'll get

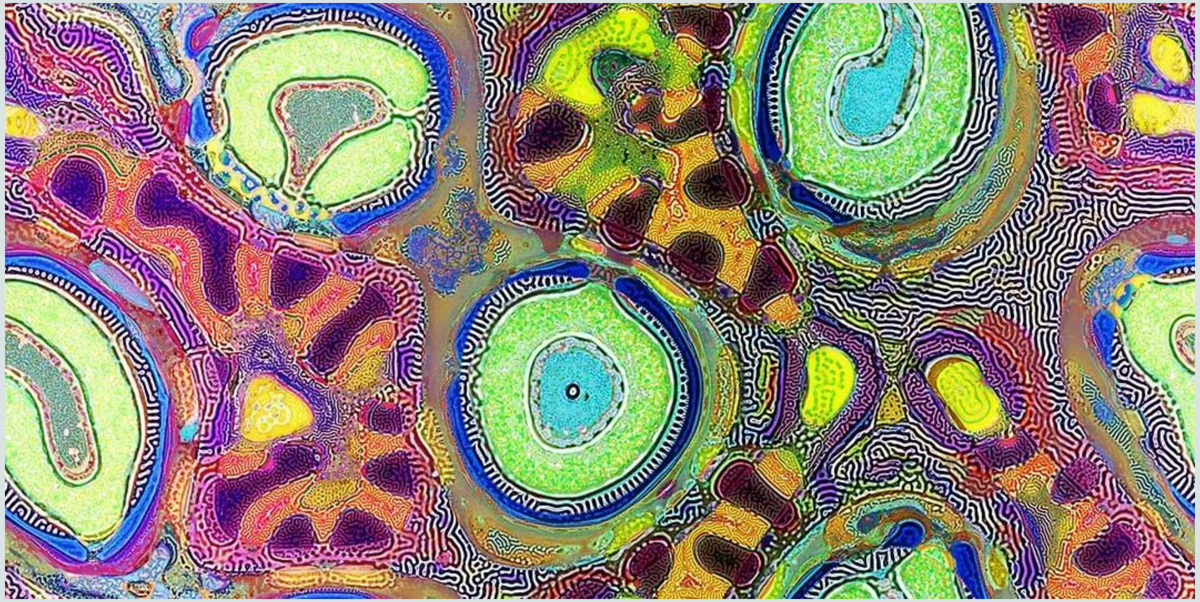
Generative Art

- Jonathan McCabe is an Australian generative artist
- Began to see the characteristic spots and stripes of Turing patterns in his artwork
- To mimic a chemical system, McCabe devised programs that used pixels in place of cells
- Program randomly assigned a number to each pixel which produces a color
- Number of pixels changes based on surrounding pixels
- First saw basic Turing patterns but eventually starting layering multiple Turing processes on top of each other – called multi-scale Turing patterns
    - Saw things like large stripes comprised of small spots

Generative Art

- Depending on what McCabe likes and does not like about his products, he can tweak an algorithm or combine pieces of different algorithms
- Many images look like iridescent fish or lizard scales, animal hide, blood vessels, or even stained tissue samples
- All of his artwork is open to interpretation



Sources

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- https://visualpde.com/mathematical-biology/gierer-meinhardt.html
- https://mason.gmu.edu/~treid5/Math447/Brusselator/
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