Mathematical model of coupled patch model

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Abstract

Coupled patch model could be used to analyze the dynamics of populations. In my presentation, I focus on the two-dimensional spatial model with two parameters. Since I choose to examine the dynamics of population, my two parameters are growth rate and dispersal rate. Growth rate refers to the dynamical change of population in the area itself, while dispersal rate refers to the dynamics of interactions between neighboring areas.

Introduction

Photo selected from Wikipedia online

In 18th century, European rabbits were first introduced to Australia and became largely widespread after that. Because they were so invasive, people started to consider them as mammalian pest as they destroyed crops and caused millions of dollars lost for famers. The government had taken several methods, such as hunting and shooting rabbit. However, the population of European rabbits did not go down until the myxoma virus was introduced to rabbit population. I was shocked by this story and began design my own model of population distribution to learn how the distribution can be understood systematically. I hope my research could be used to keep track of the dynamics of species population ahead of time before crisis occurs and preserve vulnerable population and prevent overproduction as well.

Model 1: Ricker growth model

First, I choose to base my growth function on Ricker model. The Ricker model, named after Bill Ricker is a famous and classic biological population model which gives the expected density of population Xt+1 in generation t+1 as a function of the number of individuals in the previous generation t, Xt . He first used his model in his studies of stock and recruitment in fisheries. Therefore I choose this model because it really relates to the real population change in life. By using Ricker model, the output would be a discrete time model, rather than continuous time model. Thus here is the difference between a map and an ordinary differential equation. Additionally, since I start from only considering the first parameter, growth rate, Ricker model would have no spatial dimension, but turns out to be one dimensional model. Here are the two equations for Ricker model:

$$Population\ equation: \bar{X}(t+1) = f(x(t)), \qquad (2.1)$$

$$Growth\ equation: f(x) = rxe^{-x}. \qquad (2.2)$$

The first equation illustrates the relationship between populations at successive generations, where Xt+1 represents the population at generation t + 1 and parameter t represents time. The second equation illustrates how the population grows, where x represents the population in the area that I want to analyze and parameter r is the rapidity of population growth. Now think of a population that is close to 0, meaning that initial population starts at a very small value, then e^{-x} would be nearly to 1. The growth equation would end up f(x) = rx, a simple linear function just by multiplying x by parameter r. Therefore, when the initial population starts at a small value, the growth function would be linear. If the population is tremendously large, then e^{-x} would be close to 0, thus the growth function would be the multiply of r x with a tiny number close to 0. Applicably speaking, if an area has too many individuals in the first place, the crowded environment would negatively affects the reproductive rate. This area is likely to go to extinction in the future. Next, I make use of cobweb plot since it can compare the Ricker function with the function f(x) = x, where the population experiences no change. I consider the intercepts where these two functions lines cross as fixed points. Cobweb diagram visually shows qualitative behavior of one-dimensional iterated function in dynamic system. By using Cobweb diagram, I can infer the long term behavior of the dynamic map by tracing the behavior of these two function lines.

I use the following examples to help me understand this growth function. First, I choose the growth rate r = 5, then the growth equation would be f(x) = $5xe^{-x}$. In Figure 2, the horizontal axis is X(t), population at generation t, and the vertical axis is X(t + 1), population at generation t + 1. When the slope of the growth function reaches zero, X(t + 1) reaches its maximum as the former population is 1. Next, I calculate the intercept of these two lines, which is ln(5). Therefore, when the previous population is less than ln(5), X(t+ 1) would be larger than X(t). When the previous population is above ln(5), X(t + 1) would be smaller than X(t).
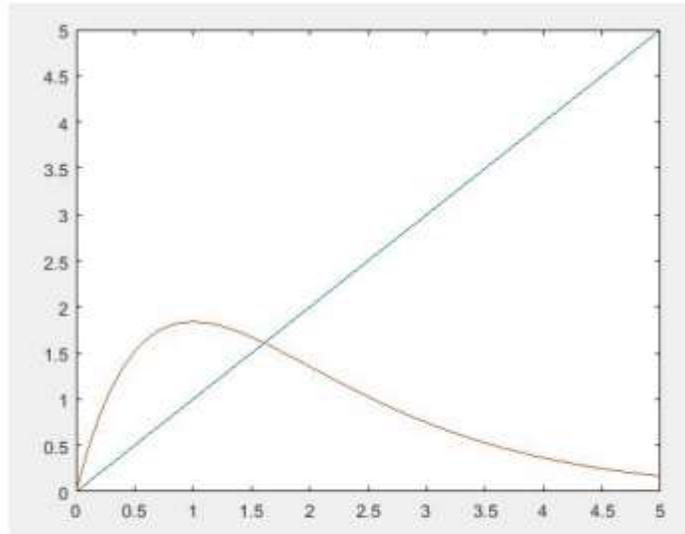
Figure 2: Cobweb plot for Ricker model when $r = 5$.

Next, setting the growth rate r = 20, then the growth equation would be f(x) =20xe^{-x} . In Figure 3, the horizontal axis is X(t), population at generation t, and the vertical axis is X(t+1), population at generation t+1. When the slope of the growth function reaches zero, X(t + 1) reaches its maximum as the former population is 1. Next, I calculate the intercept of these two lines, which is ln(20). Therefore, when the previous population is less than ln(20), X(t + 1) would be larger than X(t). When the previous population is larger than ln(20), X(t + 1) would be smaller than X(t).
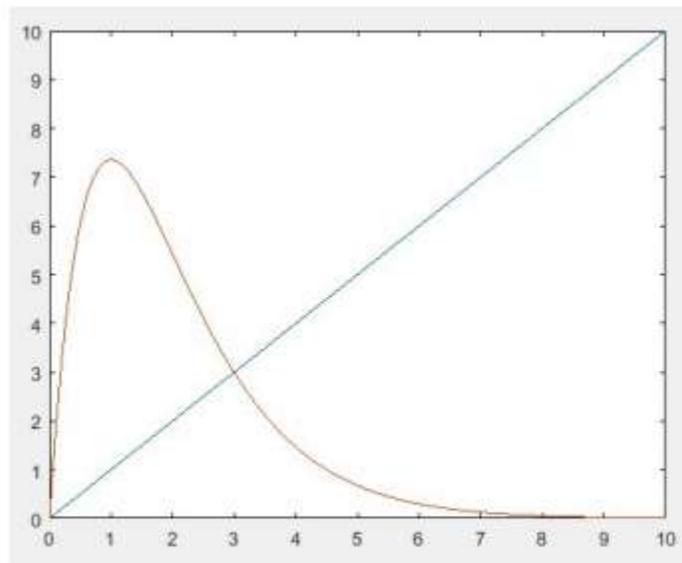

Figure 3: Cobweb plot for Ricker model when $r = 20$.

Based on these examples, I get my conclusion that the fixed point of the Cobweb diagram for Ricker model is X(t) = ln(r). Next, I draw a more detail Cobweb plot where I show every step of population changes connected by lines. The two axis stays the same for the following two figures. When the growth

rate r is 5, the fixed point is stable since the population after many iterations will approach the intercept of line of growth equation and the line of f(x) = x. In contrast, when the growth rate r is 20, the fixed point is unstable since the population after infinity iterations will move away from the intercept of the line of growth equation and the line of f(x) = x. This is how I base my understanding of the stability of one-dimensional model. Cobweb diagram shows the dynamics for one parameter combination. Now I want to move to dynamics over larger parameter space. Thus I use bifurcation diagram, which shows the values visited or approached, like fixed points, of a system as a function of a bifurcation parameter in the system. Using bifurcation diagram, I am able to examine long term behavior over different parameter combinations. In the model I created in MATLAB, I choose to overlap the population values over 100 iterations for different growth rate from 0 to 30. According to my interpretation before, when the growth rate equals to 5, the intercept of line r = 5 with the bifurcation diagram will be only one point. This agrees with conclusion that population will go to a stable fixed point. However, when the growth rate equals to 20, the intercept of line r = 20 with the bifurcation diagram will be 100 unique points. This agrees with conclusion that population will go to an unstable fixed point and dynamics will go to chaos in the end.
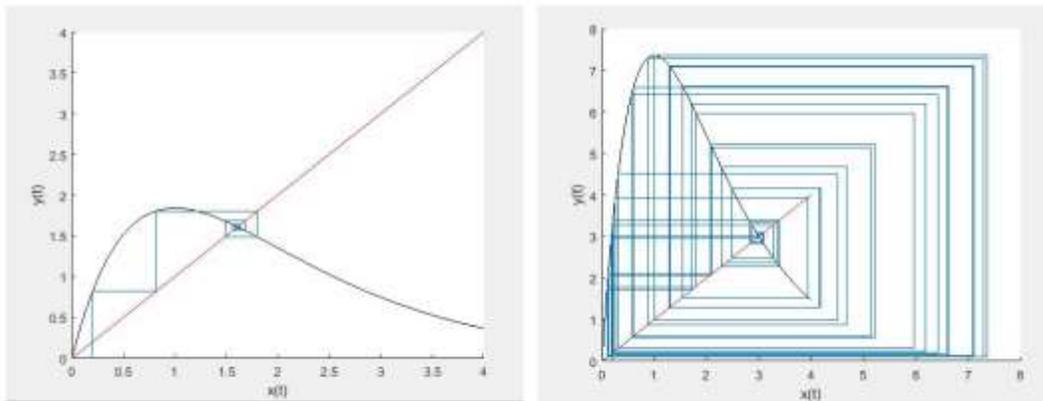

Figure 4: Detailed Cobweb plot for Ricker model. Left:$r = 5$; Right: $r = 20$.
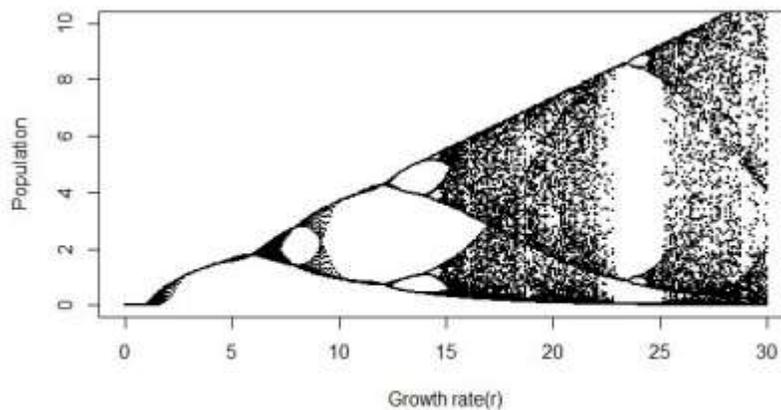

Figure 5: Bifurcation diagram for Ricker model.

Model 2: coupled patch model

Based on all these understanding of stable and unstable fixed points, I move to two dimensional map. Going back to coupled patch model, I include the second component, dispersal rate. The following equation is a simple coupled patch model equations when there is only two places and there exist dispersal behaviors between these two areas. x represents the population in area a, y represents the population in area b, d12 represents the dispersal rate from area a to area b, and d21 represents the dispersal rate from area b to area a. I assign the dispersal rate to the fraction of population moving away from its original place, thus d12 and d21 are values between 0 and 1.

The equations for population in area a and b at next generations is listed below as ⁻x and ⁻y.

$$\begin{cases} \bar{x} = (1 - d_{12})rxe^{-x} + d_{21}rye^{-y}, \\ \bar{y} = (1 - d_{21})rye^{-y} + d_{12}rxe^{-x}. \end{cases} \tag{3.1}$$

I also plot the bifurcation diagram for this simple coupled patch model. I set x = 0.5, y = 0.5, and for simplicity I assign same dispersal rate to two areas, then d12 = d21 = 0.1. Figure 6 is the bifurcation diagram for x, Figure 7 is the bifurcation diagram for y and Figure 8 is the bifurcation diagram for x + y, the total population in two areas.
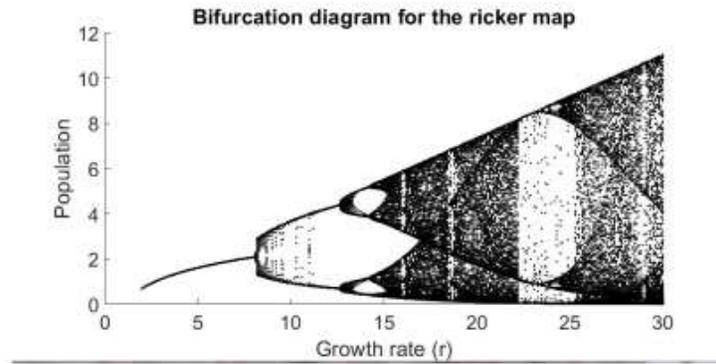
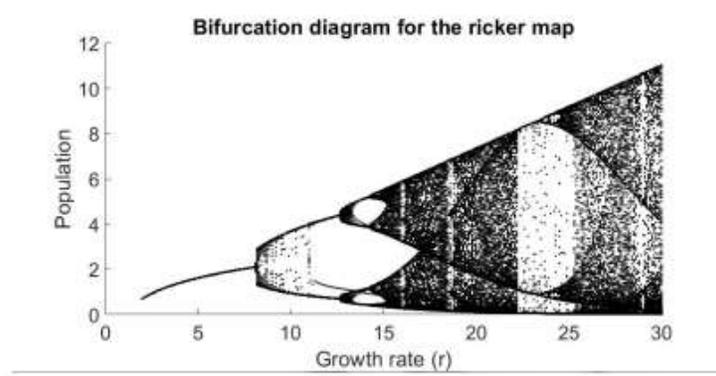Figure 6: Bifurcation diagram for simple couple patch model X.

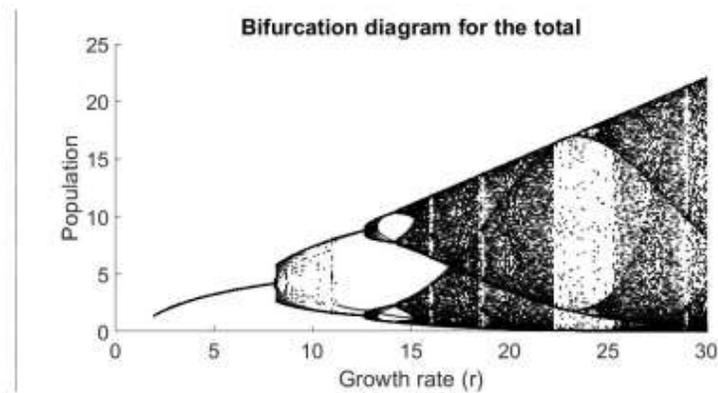Figure 7: Bifurcation diagram for simple couple patch model Y.

Figure 8: Bifurcation diagram for simple couple patch model X+Y.

Comparing these two figures, I do not see much critical differences. Thus I think bifurcation diagram cannot provide me more information about space. Therefore I change my model to a bigger model with more parameters and try to look for spatial features.

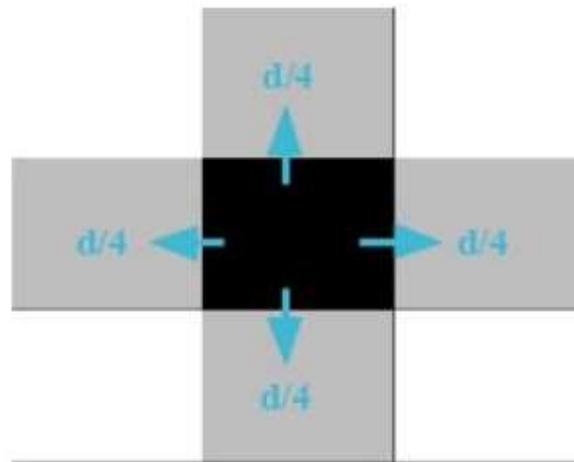## Model three: Couple Patch model in grids

Figure 9: Example provided by Professor Laura.

Again, I start from a simple model where population is only positive in the middle point of the square, and zero otherwise. The population is a random number that is near the fixed point ln(r), so the dynamics of population will be chaotic. Here the growth rate equals to 20 and the dispersal rate is 0.4. As shown in the Figure 9, instead of one neighboring area, the grid in the middle has four neighboring areas. I set the population in the middle symmetrically disperse into four directions. For color bar on the right, I set color to be blue when the density is larger than ln(20) and white when less than ln(20). In MATLAB, I create a short movie for this initial condition in 19 × 19 tables, but in order to make the movie easy to look at, I only present every other grid. Darker color represents higher density, while lighter color represents less density. Because when the initial population is high, population after one iteration will decrease much, since it will dispersal in great amount into four neighboring areas. If the population starts low, its neighbors will disperse great amount to this area. Thus, if I choose to present any grid, the movie will be always fluctuating, making it hard to look at. Since MATLAB does not allow me to run the code for zero iteration, Figure 10 is the population distribution after one iteration. This output agrees with my prediction, when the middle point starts at high density, it will have much less density after one iteration and instead its four neighboring areas would have higher density.
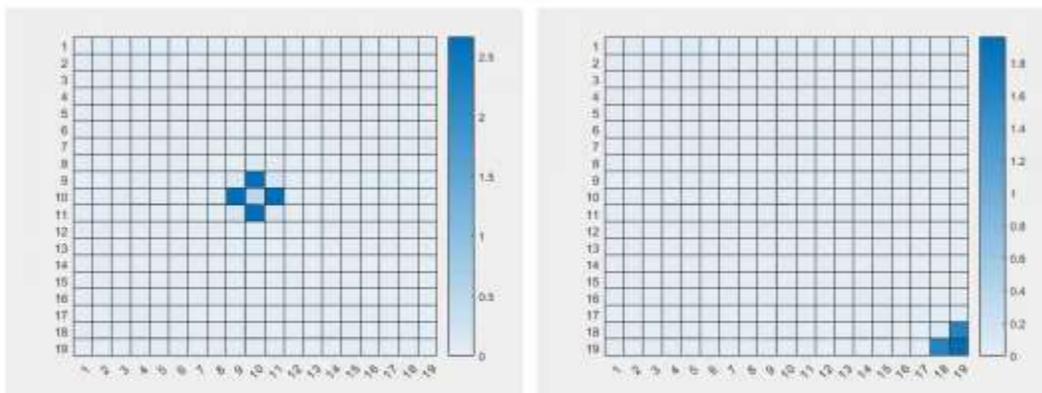

Figure 10: One time step. Left: Model one; Right: Model two

I change the initial condition to check how different dispersal scenario affects the dynamics of the model. In second model, I set population close to ln(20) only at four corners of the 19×19 tables. It catches my attention because the one time step looks like it has only one corn with population after one iteration. I rerun my codes many times and finally understand why population only appears at one corner. In my code, I set the random number between 0 and 1 and add ln(20), so the initial conditions for the four corners do not have to be the same. Therefore, it is possible that the pattern will be asymmetric and different every run. Additionally, though it looks like invasion scenario when population comes from one direction to an empty space, the other three corners population in this case will not actually go to zero, or extinct. In MATLAB, I can see exactly how much population each grid holds. I think if I can increase iteration time to test whether there would be extinction in the end.

Later, in my third model, I randomly assign population clear the fixed point to all the grids. The population distribution at one time step can be shown in Figure 11.
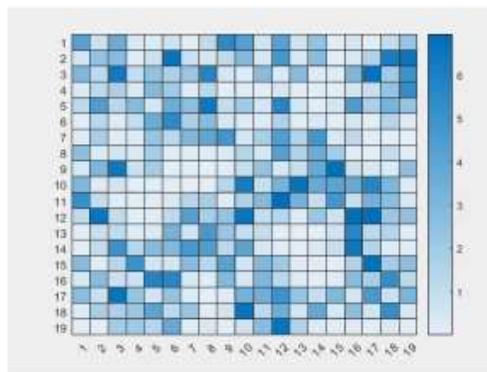

Figure 11: One time step for model three.

Visually observing the different dynamics of three models, I wonder whether different dispersal rates affect the population. From Figure 12, there are the screenshots of population distribution after 100 iterations at different dispersal rate from 0 to 0.5. Thus I generate a conclusion from these patterns that for series of images for increasing dispersal rate, the spatial scale of the patterns will also increase with it. Now I want to analyze these patterns more quantitatively, and I use the Betti number of topology to help me do that.
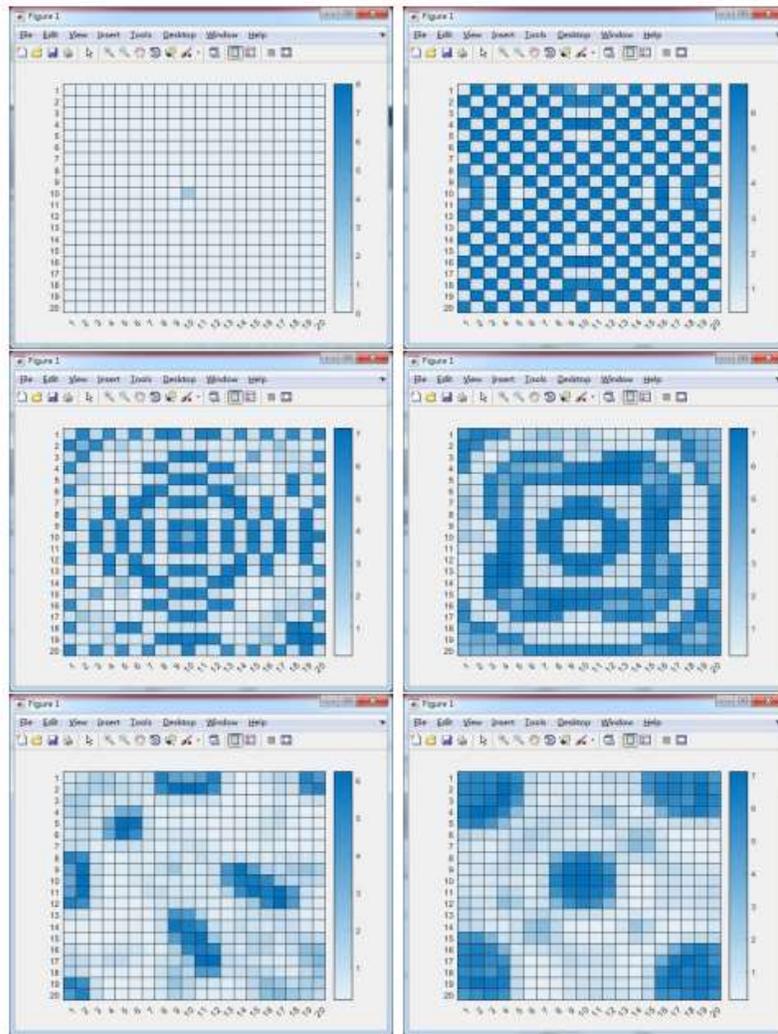
Figure 12: Population distribution after 100 iterations. Upper left: $r = 0$; Upper right: $r = 0.1$; Middle left: $r = 0.2$; Middle right: $r = 0.3$; Lower left: $r = 0.4$; Lower right: $r = 0.5$.

The definition of Betti number in topology is: kth Betti number refers to the number of k-dimensional holes on a topological surface. In my model, I only use the first and the second Betti number for spatial population time series, which respectively denotes as 0 and 1. When one calculates 0, it is just the number of connected components in the graph. When one calculates 1, it is just the number of holes in the graph. I list two examples, given by Professor Laura, to illustrate how to calculate when one calculates 0 and 1. In left panel of Figure 13, 0 would be 3 and 1 would be 1. In right panel of Figure 13, 0 would be 5 and 1 would be 2.

Since I want to know the dynamics of these three models with different initial conditions, I generate 2 time series plot for each model, one with dispersal rate 0.1 and one with dispersal rate 0.4 and the growth rate is always 20. When I generate the time series plot, I make images binary for betti number calculation because MATLAB does not take grids as an input. In the first row and second row panels of Figure 14, it is

obvious that 0 is much less than 1 when the dispersal rate is 0.1, meaning the pattern looks like one connect component with many holes in it. However, when the dispersal rate increases to 0.4, 0 tends to be larger than 1, meaning the pattern looks like a checkerboard. Thus I get my conclusion that with lower dispersal rate, betti numbers dynamics tends to be periodic, flipping between two points and settle down to one point in the end and with higher dispersal rate, betti number dynamics tends to be more chaotic, which ends up fluctuating. In the last row panels of Figure 15, I think because the initial condition is different from the other two models, the dynamics of the third model is more complex and chaotic, which leads to my later conclusion.
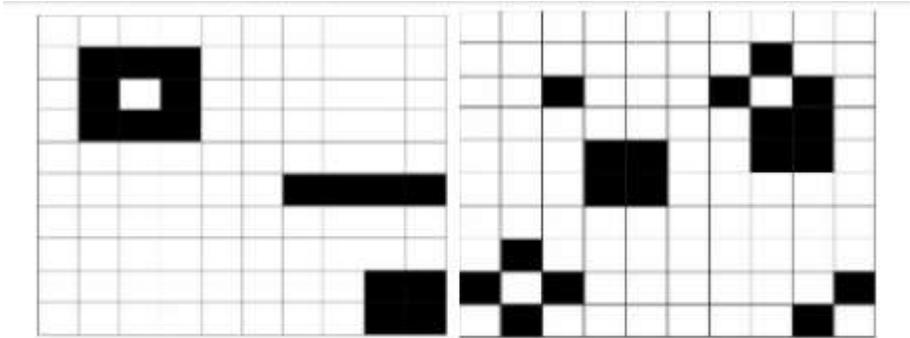

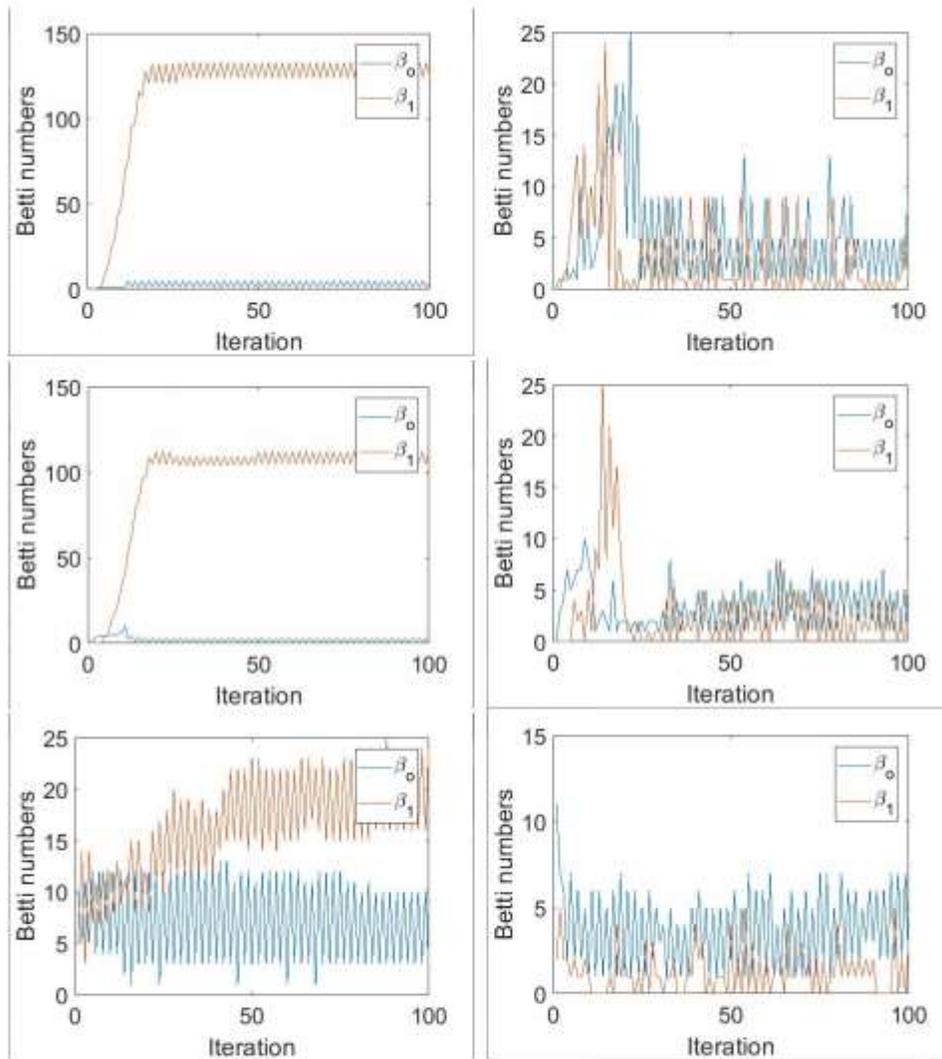Figure 13: Betti number example provided by Laura.

Figure 14: Time series plot of betti number. The first row is for model one, the second row is for model two and the third row is for model three. Left column: $d = 0.1$; Right column: $r = 0.4$.

Figure 15: Time series plot of betti number. The third row is for model three. Left: d=0.1;

Right : r=0.4

Conclusion:

Based on all the model I analyze, I think smaller dispersal rate will likely lead to periodic behavior of population while larger dispersal rate will lead to more chaotic behavior of population. Additionally, besides dispersal rate, the initial condition matters to the dynamics of model.

Further work and reflections:

Thus for my further steps, I would move to more complex dispersal scenario. I want to distribute the population based on its own choice. Rather than setting the dispersal rate, the animals or plants can make their decisions on where to disperse by comparing its neighboring areas and choose its preferences. I am also thinking about creating network between indirect neighbors, making the scenario more complicated. After this step, I want to use actual data from literature to test the usefulness of my conclusion and work with GIS data to move my research into a more applicable direction.

Also, I got some great comments and ideas from professor CK and my classmates. In the grid model, I set the directions of population distribution into 4 direction just for simplicity. As suggested by classmates and professor CK, a hexagonal grid graph or octagonal grid graph would increase the credibility of my model. Some suggest me considering the effect of predators and hunting. I really appreciate these comments but I think adding too many factors to a model would compromise our understanding on what factors would influence this model most. Therefore, I think I should either consider these factors after I move to more difficult scenario or use different dispersal rate to represent the underneath predator factors. Lastly, I think I should have a clearer goal by focusing on one topic one time. I would pay attention to all these problems above for my further steps. Overall, I think my project could be applied to real life issue, for instance the spread of disease, and I would continue working on it.

As a reflection of my performance of the presentation, I think I explain every concepts clearly by adding additional slide of easy example. However, just as my classmates said, I should illustrate more about the reason why I use certain definition. Otherwise, some unnecessary concepts should be omitted to make it easier for audiences to understand. I should have also taken better control of time, thus leaving more time for open questions. Interestingly, my classmates recommend one computer game for me, the Conway's game of life. This game has the similar life and death rules, which I think it would be interesting to compare the result of my simulation and the result of the game.