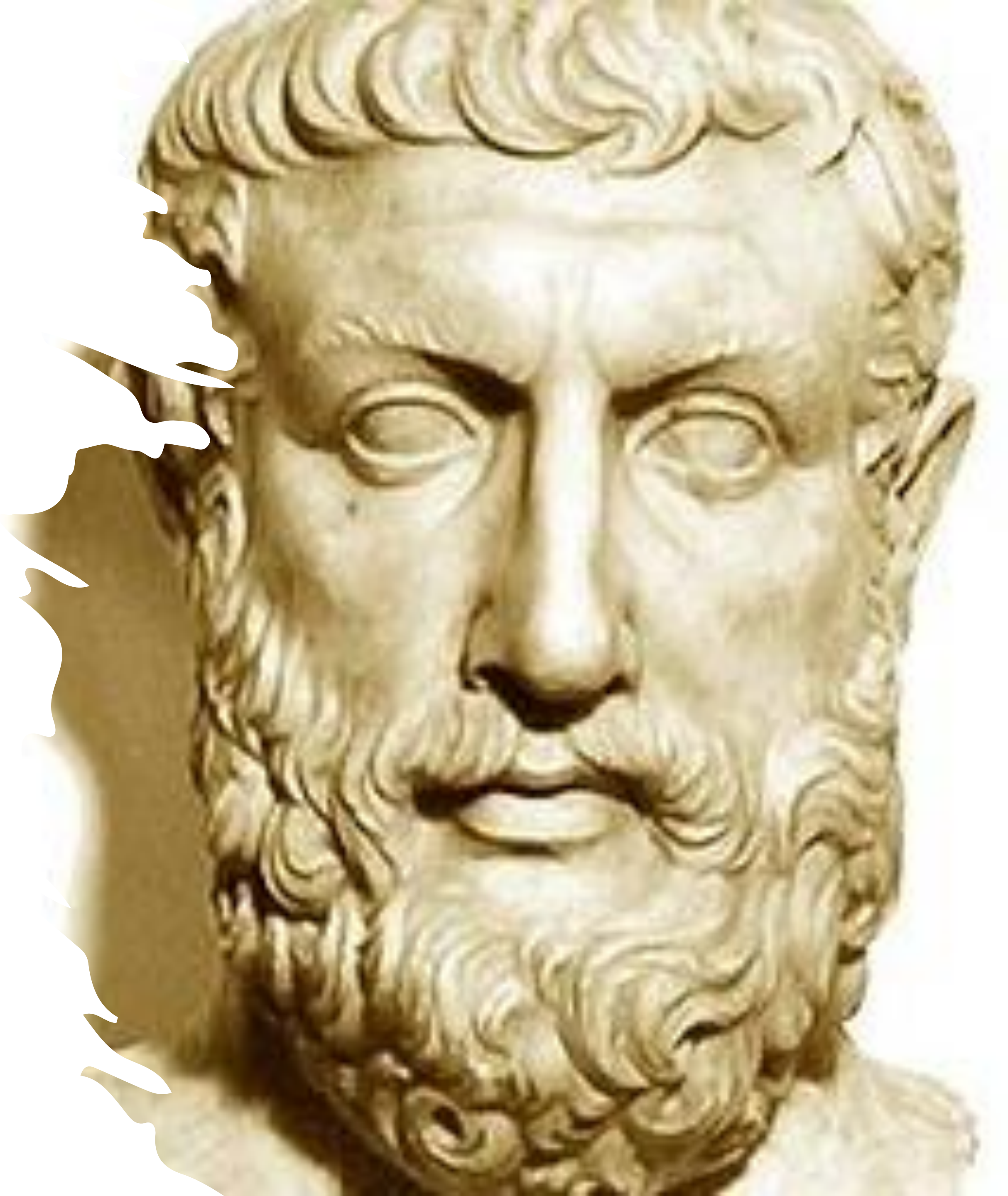
Zeno's Paradox
Calculus & Analysis

JARED JONES

The Eleatic School

- "Come now, I will tell thee … the only two ways of inquiry that can be thought of. The first, namely, that *It is*, and that it is impossible for anything not to be, is the way of. conviction, for truth is its companion. The other, namely, that *It is not*, and that something must needs not be, - that, I tell thee, is a wholly untrustworthy path. For you cannot know what is not - that is impossible - nor utter it" (Parmenides, 2).
  - Parmenides – Born 515 BCE
  - Zeno – Born 495 BCE



Elements of Eleatic Thought

In effect, being is, nonbeing is not. Whenever you think of something ‘not being’, then, what are you thinking of? To the extent you are thinking nonbeing, you are not thinking of anything, and yet you act as if you are thinking of something. There is a fundamental error involved in ever thinking negation or nonbeing: We must treat ‘what is not’ as something which *is*.

- The results of this for the Eleatic School are profound: Motion, change, difference, and the like must be relegated to the status of mere semblances which, in truth, can be proven not to be real at all.

Zeno's Paradox

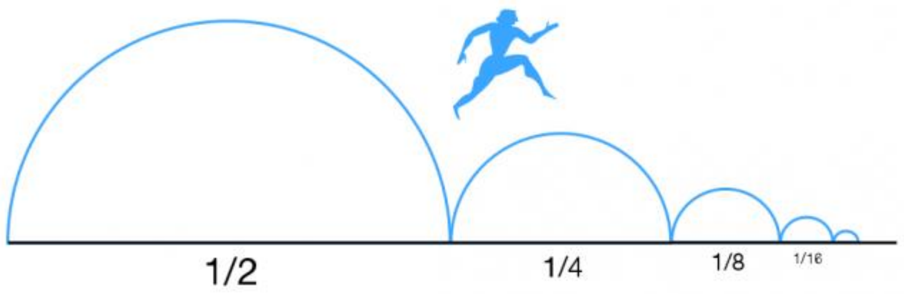
The Role of the Paradoxes

ZENO'S PARADOXES ARE DESIGNED TO BE FURTHER PROOF OF THESE IMPLICATIONS OF PARMENIDES'S PHILOSOPHY.

THESE PARADOXES REVEAL SUPPOSED INCOHERENCIES IN MOTION, MULTIPLICITY, AND THE LIKE THAT ARISE FROM THE CONSTITUTIVE ERROR IN THINKING NONBEING OR NEGATION.

“The Dichotomy”

“The first asserts the nonexistence of motion on the ground that that which is in locomotion must arrive at the halfway stage before it arrives at the goal” (Aristotle, 293b11).



Mathematical Solutions to The Dichotomy: I. Formal Logic

One way to solve The Dichotomy is what philosophers call a ‘Moorean Shift’. Zeno’s argument could be formalized as follows:

ORIGINAL	***MOOREAN SHIFT***
1. $P(m)$.	1. $P(m)$.
2. $P(m) \rightarrow \big(F(j) \land M(j)\big)$.	2. $P(m) \rightarrow \big(F(j) \land M(j)\big)$.
3. $\forall x \in T, M(x) \rightarrow \neg F(x)$.	3. $\neg\big(F(j) \land \neg F(j)\big)$.
4. $\therefore F(j) \land \neg F(j)$.	4. $\therefore \neg\big(\forall x \in T, M(x) \rightarrow \neg F(x)\big)$.
5. $\neg\big(F(j) \land \neg F(j)\big)$.	
6. $\therefore \neg P(m)$.	

Key: P(x): “x is possible”. m: “motion”. F(x): “x is completed in a finite amount of time”. M(x): “x requires infinitely many trips be completed”. j: “the journey”. T: “the set of all trips”.

Discussion

1. WHAT DO YOU THINK ABOUT MOOREAN SHIFTS?
2. IS THIS A SUCCESSFUL REFUTATION OF THE DICHOTOMY?

Mathematical Solutions to The Dichotomy: II. Aristotle's Solution I

Aristotle points out that we must divide the time along with the distance traveled.

- If my speed is 1distance unit per 1 time unit, then we can directly convert one into the other:

$$x\, u_t \left(1 \frac{u_d}{u_t} \right) = x\, u_d$$

- Accordingly, it will take ½ a time unit to go ½ the distance. Likewise, ¼ to go ¼ the distance. And so on.

Mathematical Solutions to The Dichotomy: II. Aristotle's Solution II

In general, the time or distance passed (they are equal) at the nth trip is given by this equation:

$$t_n = {(2^n - 1)}/{2^n} = 1 - ({1}/{2^n})$$

But as you can see, *this quantity can never be greater than 1*. In other words, 1 is an *upper bound* for the terms in this sequence.

A Complication to Aristotle's Solution

The Dichotomy can be read in two ways: a more sophisticated way than saying

1. "Infinitely many finite trips must take an infinite amount of time."
2. "Infinitely many finite trips *cannot be completed* in a finite amount of time."

It seems, however, that Aristotle's first reply here answers the first interpretation but not the second one.

Discussion

1. WHAT DO YOU THINK ABOUT ARISTOTLE’S SOLUTION?
2. DOES THIS REPLY ON BEHALF OF ZENO HAVE A GOOD POINT?
3. DOES ARISTOTLE HAVE A GOOD REPLY TO THIS OBJECTION?

Mathematical Solutions to The Dichotomy: III. The Geometric Series

Another way to approach the problem, then, would not be from the bounds of these finite totals but instead directly as an infinite sum:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n}$$

We learned in calculus that this is a *geometric series*, and we can prove using analysis that, since $|1/2|<1$, the following holds:

$$\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n}=\frac{1/2}{1-1/2}=1.$$

This result, of course, is just what we would expect, certainly not an infinite amount of time.

Concluding Discussion

1. HOW DOES THIS REPLY DIFFER FROM THE ONE ARISTOTLE GAVE?
2. COULD ZENO REPLY ALONG THE SAME LINES AS BEFORE?

The Adequacy of this Interpretation I

But does this argument solve the problem? Zeno might not be convinced. Consider the definition of the convergence of an infinite series in analysis:

- Def 1. An infinite series *converges* to a if the sequence of partial sums converges to a.

So, the infinite sum expression is really something like a 'shorthand' for the *limit* of the sequence

$$1/2, 3/4, 7/8, 15/16, \ldots$$

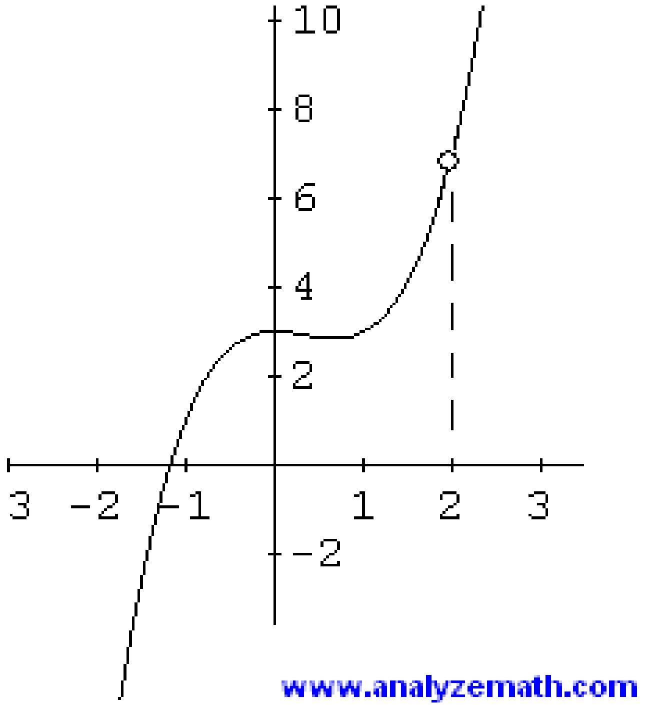
which we saw before in Aristotle's solution.

The Adequacy of this Interpretation II

The trouble begins when we parse what the convergence of a sequence means. This is the definition in analysis:

- Def 2. A sequence (a_n) converges to a if $\forall \varepsilon \in \mathbb{R}^{+}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N \rightarrow |a_n - a| < \varepsilon$.

Zeno's might be answer us: 'Yes, I accept you get closer and closer to 1 time unit, closer and closer to 1 distance unit. This approach even satisfies your 'convergence' definition. *But it remains true that your 'approach' <u>never</u> actually arrives at the destination*. And that is what is important here.'



A Familiar Problem from Calculus

The difference Zeno would be pointing out here is not an idle one. It is familiar from limit questions in calculus.

I could form a sequence of the function on the left by

$$(a_n) = \left(f\left(2 - \frac{1}{n}\right)\right) = f(1), f\left(\frac{3}{2}\right), f\left(\frac{7}{4}\right), \ldots$$

This sequence converges to about 7. But the function never actually takes on the value of 7.

Discussion

1. HOW DOES THIS OBJECTION DIFFER FROM THE ONE TO ARISTOTLE?
2. WOULD ZENO HAVE A GOOD POINT IF HE WERE TO REPLY ALONG THESE LINES?

Mathematical Solutions to The Dichotomy: Aristotle's Second Solution

Aristotle's second objection, we might say, is one fundamentally about modeling. Zeno uses the wrong basic mathematical model for the motion.

He says that motion in time proceeds along a '*continuous*' line. He says that continuous motion contains these discrete steps 'potentially', but it is not the actual journey being made.

- In modern mathematical terms, we should be modeling the time passed, and the motion in general, using a *continuous function*.
- From this function, we can see that the infinite series is contained in it; that is, if $f(x) = x$ where $x \in [0,1]$ is the distance traveled, then we have this:

$$\left\{ {(2^n - 1)}/{2^n} : n \in \mathbb{N} \right\} \subseteq f([0,1]) = \{f(x) : x \in [0,1]\}$$

- So, if we simply model it as a continuous function, we can see that the runner has completed all the trips that Zeno wanted, and he has done the entire trip.

Concluding Discussion

1. DOES THIS WORK AS A SOLUTION TO ZENO'S PARADOX? OR DOES IT JUST AVOID THE QUESTION?
2. DID WE IDENTIFY WHERE HE WENT WRONG IN FRAMING THE DICHOTOMY? WHERE DID HE?
3. ANY GENERAL OR FURTHER THOUGHTS?

Bibliography

- Aristotle, ‘Physics’, W.D. Ross (trans), in *The Complete Works of Aristotle*, J. Barnes (ed.), Princeton: Princeton University Press, 1984.
- Huggett, Nick, "Zeno’s Paradoxes", *The Stanford Encyclopedia of Philosophy* (Winter 2019 Edition), Edward N. Zalta (ed.).
- Parmenides. ‘Fragments and Commentary’,John Bunet (trans.), Arthur Fairbanks (trans.), and H.H. Joachim (trans), in *Early Greek Philosophy*.