Minimax Algorithm and Its Application

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Abstract

Minimax is a famous backtracking algorithm that is widely used in decision making and game theory. Von Neumann developed first Minimax theorem in 1928, and since then, game theory began to flourish. The Minimax theorem is usually used to search for the optimal move for a player, with an assumption that your opponent also plays optimally. In this paper, I discussed the mathematical connection between optimal move found by Minimax theorem and its application in real-life, specifically, in the rock-paper-scissor. This paper also discussed some interesting application of Minimax theorem and its significance in real world.

1 Introduction

The fundamental theorem of game theory, Minimax Theorem, was first proved by John von Neumann is 1928. It states that any finite, zero-sum, two players game has optimal mixed strategies. Player would either play as maximizer who always tries to get the highest score possible, or as a minimizer who tries to get the lowest score possible, which is minimizing his/her own losses. Finally, two players would reach a subtle balance, which is the optimal point for both of them. This approach of problems defines a new relationship in the competition and cooperation, and Minimax Theorem became a new guide of economic behavior and was widely used in many areas such as economics, mathematics, psychology and sociology.

Minimax Theorem has two assumptions. First, it can only be applied to the two players zero-sum game. The zero-sum game is a situation in which one person wins only by causing the other person to lose, so the total utility is divided among the player and the payoff is always balanced. The common two player based zero-sum games include chess, tic-tac-toe and rock-paper-scissor. The second assumption is that, both players are expected to play optimally, i.e., both players will play in such a way to maximize their own chance of winning.

This paper examines what is the best strategy to win the popular zero-sum game, the rock-paper-scissor. This topic is of great interest as finding the optimal move both in games and real-life can be hard. People usually make their individual decisions based on motivation and emotion.

Yet, making a rational and optimal choice requires more than just inner feelings; it requires strategy and careful examination about not only advantages but also limitations. Thus, I hope by discussion this topic, this paper cold shed more light on choosing best strategy and better guide readers in the decision-making process.

2 Problem Introduction

Why Rock-Paper-Scissors is a classic example of zero-sum game? Recall that, in rock-paper-scissors, two players choose one alternative among rock, paper, or scissors simultaneously. The rule is that rock beats scissors, scissors beats paper, and paper beats rock. Most importantly, rock-paper-scissors is a zero-sum game. This could be specified by a payoff matrix. First choosing row to represent payoff for player A, then choose column to represent payoff for player B. The following matrix describes the payoffs in the Rock-Paper-Scissors game:

Player A / Player B	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Therefore, Rock-Paper-Scissors is two players based zero sum game. So, the essential question is, what is the best strategy to win the Rock-Paper-Scissors? Or, does there exist an optimal move for both players? According to the Minimax theorem, the answer is yes, there does exist an optimal strategy.

We could re-formulate the payoff matrix above into the equations. The payoff matrix is a $m \times n$ matrix A, and the row player A picks strategy $i \in \{1, \ldots, m\}$ with probability x_i, the column player B picks strategy j$\in \{1, \ldots, n\}$ with probability y_j, and player A pays player B a_{ij}. So, if player A uses random strategy x and player B uses y, then expected payoff from row player A to column player B is:

$$\sum_i \sum_j x_i a_{ij} y_j = x^T A y$$

For column player B, suppose he adopt strategy y. Then row player A's best strategy is to use x that minimizes the expected payoff: $\min x^T Ay$. So, the column player B should choose the y which maximizes his probability to win: $\max \min x^T Ay$.

3 Finding Best Strategy

Specifically, two players, the maximizer and minimizer, will adopt different strategies. The maximizer will choose maximin strategy to maximizes one's worst-case payoff. By adopting maximin strategy, the minimum amount of payoff is guaranteed. In contrast, the minimizer will choose minimax strategy to minimize opponent's best-case payoff.

Maximin side

Let x_i= the probability that player A chooses action *i*, for *i* ∈ {Rock, Paper, Scissors}. Then Player A's maximin strategy can be found by solving the following optimization model:

$$Max \quad \min\{x_2 - x_3, x_3 - x_1, x_1 - x_2\}$$

$$subject\ to\ x_1 + x_2 + x_3 = 1$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

This model is not linear currently, so we need to standardize it by play the linearization tricks:

$$Max \quad \text{z}$$

$$subject\ to\ z \leq x_2 - x_3$$

$$subject\ to\ z \leq x_3 - x_1$$

$$subject\ to\ z \leq x_1 - x_2$$

$$subject\ to\ x_1 + x_2 + x_3 = 1$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Now the model is linear, and we want to reorganize it, so all the decision variables are on the left-hand side of the constraints, and all constant are on the right-hand side:

$$Max \quad \text{z}$$

$$subject\ to\ z - x_2 + x_3 \leq 0$$

$$subject\ to\ z + x_1 - x_3 \leq 0$$

$$subject\ to\ z - x_1 + x_2 \leq 0$$

$$subject\ to\ x_1 + x_2 + x_3 = 1$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Minimax side

Let v_i= the probability that player B chooses action *j*, for *j* ∈ {Rock, Paper, Scissors}. Then Player B's minimax strategy can be found by solving the following optimization model:

$$Min \quad \max\{-v_2 + v_3, v_1 - v_3, v_2 - v_1\}$$

$$subject\ to\ v_1 + v_2 + v_3 = 1$$

$$v_1 \geq 0, v_2 \geq 0, v_3 \geq 0$$

This model is not linear currently, so we need to standardize it by play the linearization tricks as we did for the Maximin model:

$$Minmize \quad w$$

$$subject\ to\ w \geq\ v_3\ - v_2$$

$$subject\ to\ w \geq v_1 - v_3$$

$$subject\ to\ w \geq v_2 - v_1$$

$$subject\ to\ v_1 + v_2 + v_3 = 1$$

$$v_1 \geq 0, v_2 \geq 0, v \geq 0$$

So now we successfully constructed the linear formulations for each player, and we could go back to go maximin model first. Our goal is to solve the $x_1, x_2\ and\ x_3$ for player A and $v_1, v_2\ and\ v_3$ for player B. Usually, a technique to solve the linear model is to first solve the dual problem. Therefore, the next step is to convert the original primal problem to the dual problem.

$Max \quad z$	$Min \quad w$
$subject\ to\ z - x_2 + x_3 \leq 0$	$subject\ to\ w \geq\ v_3\ - v_2$
$subject\ to\ z + x_1 - x_3 \leq 0$	$subject\ to\ w \geq v_1 - v_3$
$subject\ to\ z - x_1 + x_2 \leq 0$	$subject\ to\ w \geq v_2 - v_1$
$subject\ to\ x_1 + x_2 + x_3 = 1$	$subject\ to\ v_1 + v_2 + v_3 = 1$
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$	$v_1 \geq 0, v_2 \geq 0, v \geq 0$

It is obvious, now, that through the comparing and contrasting the dual problem of player A and primal problem of player B, we found they are exactly the same. This connection between solution of player A's model and solution of Player B's model is the strong duality theory. The Strong Duality Theory states that, if either Primal or Dual has a finite optimal value, then so does the other, and optimal solutions to both Primal and Dual exist and are the same. Therefore, in this case solution of Maximin model is just the same as the solution of Minimax model.

By using AMPL, the solution of model could be solved easily, and it shows that $x_1 = x_2 = x_3 = \frac{1}{3}$, same for the solution of Minimax model: $v_1 = v_2 = v_3 = \frac{1}{3}$.

```
var x1 >= 0;
var x2 >= 0;
var x3 >= 0;
var w;

minimize objective: w;
subject to cons1: x1+x2+x3=1;
subject to cons2: x2-x3+w>=0;
subject to cons3: x3-x1+w>=0;
subject to cons4: x1-x2+w>=0;
```

```
CPLEX 12.8.0.0: optimal solution; objective 0
3 dual simplex iterations (2 in phase I)
ampl: display x1;
x1 = 0.333333

ampl: display x2;
x2 = 0.333333

ampl: display x3;
x3 = 0.333333
```

Interesting, the optimal strategy is actually to play each alternative randomly, and this is exactly why when playing Rock-Paper-Scissors, we need to vary our strategy in order to mimic the randomness.

4 Limitation

The optimal strategy, however, is usually hard to achieve in reality. There are a few reasons we might doubt the practical applicability of the Minimax Theorem. Firstly, in order for a mixed strategy solution to be played, each player must choose a strategy corresponding to the opponent's strategy. However, it is quite counterintuitive for each player to choose strategy based on their opponent's payoff. Secondly, since people are not genuine random number generators, it is almost impossible to truly randomize our choices. Even worse, our failed attempts at randomization may leads to predictability that could be utilized by opponents. Thirdly, what if the opponent just doesn't play with equilibrium strategy? In this case, it makes no sense for play the mixed strategy as you need to change your strategy accordingly. In the end, neither of two players adopt mixed strategy. Consequely, the equilibrium is highly unstable.

5 Reflection

Even the Minimax Theorem is not without its own limitations, it still provides a lot of insight into decision-making and it gave birth to the theory of duality. It has several contributions: it guides decision making process and has a strong impact on the social sciences including economics, political science and psychology. Furthermore, the Minimax Theorem provides us with the tools to analyze payoffs of various situation. The motivation for me to choose this topic is because I hope to further the study of Minimax theorem and better guide people in decision-making process. I hope I could learn more about it through studying *Judgment and decision-theory* next semester.

Reference

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